11 April 2024

For a simplicial complex $K$, the **Euler Characteristic** of $K$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \, \#\{i\text{-dimensional simplices in } K\}$$

chi

number of items in this set

Think: (num. vertices) − (num. edges) + (num. faces) − ...

For a 2-dimensional simplicial complex:

$$\chi = \begin{pmatrix}\text{number of} \\ \text{connected} \\ \text{components}\end{pmatrix} - \begin{pmatrix}\text{number} \\ \text{of} \\ \text{holes}\end{pmatrix}$$

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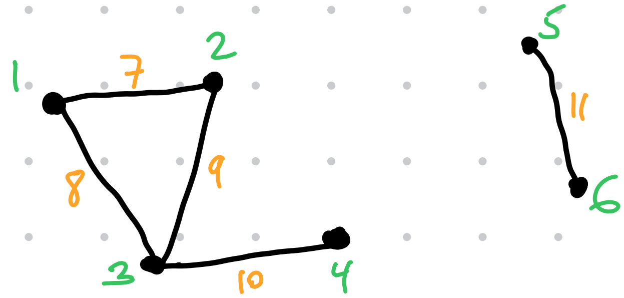
## How do we compute "holes" in a simplicial complex?

Example:

K:




Define boundary matrices that encode which simplices are in the boundary of which other simplices.

"Boundary" of an edge: endpoints

columns labeled by edges in $K$

rows labeled by vertices

$$\begin{array}{c|c} & \begin{matrix} 7 & 8 & 9 & 10 & 11 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \partial_1$$

Mod 2 arithmetic:

$1+1=0$

add colums 7 and 8:

$$\underset{7}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \underset{8}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \underset{9}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

With mod 2 arithmetic, colum 9 is a linear combination of 7 and 8.

Mod 2: $\operatorname{rank}(\partial_1) = 4$

Formula for $\beta_0$ = number of connected components)

$$\beta_1 = \begin{pmatrix} \text{number of} \\ \text{vertices} \end{pmatrix} - \operatorname{rank}(\partial_1)$$

$$= 6 - 4 = 2$$