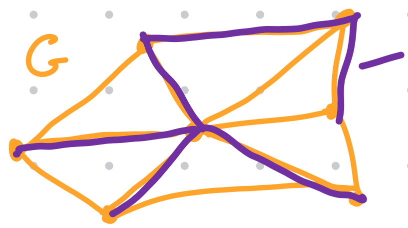
19 March 2024

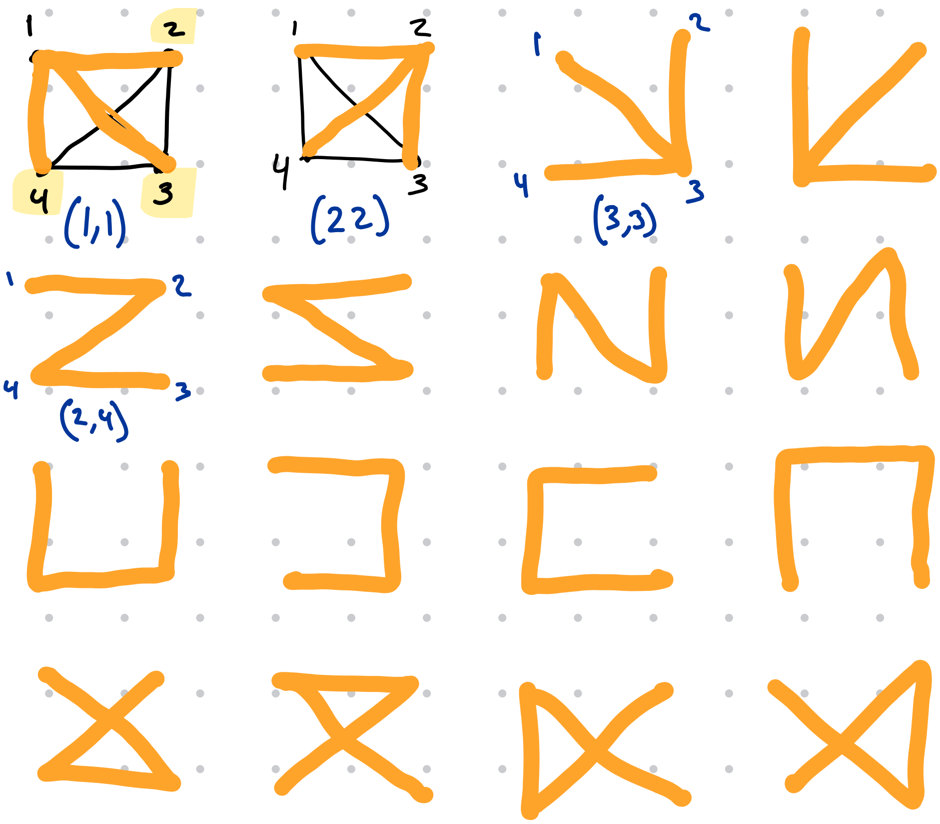
SPANNING TREE. Given a graph G, a spanning tree is a subgraph of G that includes all vertices of G and has no cycles.

example:



Q: How many spanning trees are there in the complete graph on n vertices? K_n

eg. $n=4$:



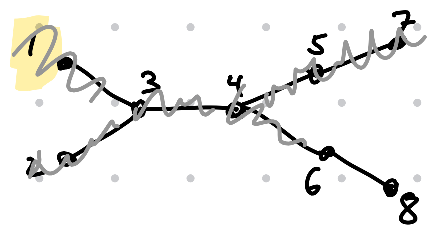
16

3

Encoding a tree with vertices labled $\{1, 2, 3, \ldots, n\}$
as a sequence of $n-2$ integers, each $1, 2, 3, \ldots, n$:

example:

$n = 8$



3, 3, 4, 5, 4, 6

Find the leaf with lowest index. What is it connected to?

$1 \to 3$

Next leaf with lowest index: $2 \to 3$

Next: $3 \to 4$

etc.

We can reconstruct the tree from the list 3, 3, 4, 5, 4, 6,

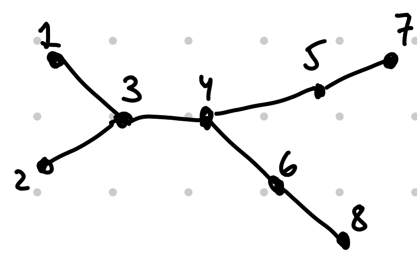
We know: $n = 8$ vertices.

$$P = (\cancel{3}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{4}, 6) \qquad V = \{\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, 6, \cancel{7}, 8\}$$

Let v be the smallest element of V not in P.

Connect 1 to the smallest element in P:

Repeat



For more information, see "Counting Spanning Trees" by Bang Ye Wu and Kun-Mao Chao, https://www.csie.ntu.edu.tw/~kmchao/tree07spr/counting.pdf

This is an example of the Prüfer Encoding of a labeled tree.

$\{$Set of labeled trees on n vertices$\}$ $\longleftrightarrow$ $\{$Set of sequences of length $n-2$ of n symbols$\}$

sizes: n^{n-2}

For a collection of n cities: are there more TSP tours or more spanning trees?

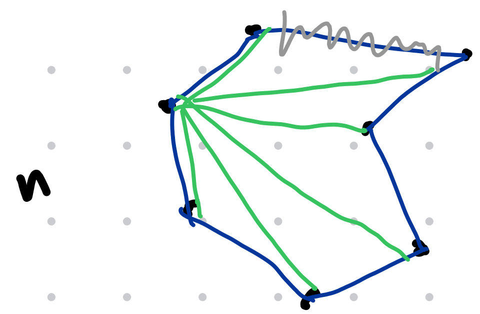
Spanning trees: n^{n-2}

TSP tours: $(n-1)! \sim \sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}$

$\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)\left(\frac{n-1}{e}\right)^{n-2}$

Stirling's Formula:

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$



→ There are more spanning trees than TSP tours!

→ Finding the minimal TSP tour is hard

→ Is it hard to find the minimal spanning tree? NO!

KRUSKAL'S ALGORITHM:

Polynomial Time

Compute all edge lengths. $O(n^2)$

Sort all possible edges by length. – $O(n \cdot \log n)$

Initialize MST = { }.

$O(n^2)$ { Loop over all edges from shortest to longest:
   If adding the current edge to MST does not make a loop, then add the edge to MST.

MINIMUM SPANNING TREE: spanning tree with smallest total edge length