The Hausdorff Property

MATH 348

1. Which of the following spaces are Hausdorff spaces?

(a) $[0, 1]$ with the subspace topology from $\mathbb{R}$

(b) $\mathbb{R}^n$ with the standard topology

(c) $\{a, b\}$ with the indiscrete topology

(d) Any set X with the discrete topology

(e) Set $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$

2. Let T be a topological space. Suppose A is a subset of X such that for every $x \in A$ there exists open set U such that $x \in U \subset A$. Prove A is open.

3. Prove that if T is a Hausdorff space, then every single-point subset of T is closed.

4. If T is a topological space and $f : T \to T$ is a continuous map, then the **fixed-point set** of f is defined $\text{Fix}(f) = \{x \in T \mid f(x) = x\}$.

Prove that if T is a Hausdorff space then $\text{Fix}(f)$ is a closed subset of T.

5. For integers $a \neq 0$ and b, let $S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$.

(a) Show that the collection $\{S(a, b) \mid a, b \in \mathbb{Z}, a \neq 0\}$ is the basis for a topology on $\mathbb{Z}$. This topology is called the **arithmetic sequence topology**.

(b) Show that the arithmetic sequence topology is Hausdorff.

(c) Show that the basis elements are both open and closed in this topology.

(d) Use topology to prove the infinitude of primes as follows:
Let $Q = \bigcup_{p \text{ prime}} S(p, 0)$. Assume that there are only finitely many primes, and explain why this implies that Q is closed.
Then $\mathbb{Z} - Q$ must be open. What integers are elements of $\mathbb{Z} - Q$? Why is this a contradiction?