Bessel's Equation

Math 330

The following ODE is is known as **Bessel's differential equation of order** $m > 0$:

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0$$

If m is not an integer, then Bessel's equation of order m has two linearly independent solutions denoted $J_m(z)$ and $J_{-m}(z)$, with

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{m+2k}}{2^{m+2k} k!\, \Gamma(m+k+1)},$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is the Gamma function. The function $J_m(z)$ is the **Bessel function of the first kind of order** m.

If m is an integer, then $J_{-m}(z) = (-1)^m J_m(z)$, so these two solutions are not linearly independent. A second solution linearly independent to $J_m(z)$ is given by

$$Y_m(z) = \lim_{\alpha \to m} \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)}$$

the **Bessel function of the second kind of order** m.

1. Plot the functions $J_m(z)$ and $Y_m(z)$ for various m to see the shape of their graphs. What do you observe about these functions?

2. Suppose we impose the boundary conditions $|f(0)| < \infty$ and $f(a) = 0$. The condition at $r = 0$ is known as a *singular* boundary condition, and the other boundary condition is imposed at $r = a$. Which of the Bessel functions can satisfy these boundary conditions? What eigenvalues do these boundary conditions determine?

3. The Bessel functions of the first kind satisfy the following orthogonality property:

$$\int_0^1 J_m\left(\sqrt{\lambda_{mp}}\, r\right) J_m\left(\sqrt{\lambda_{mq}}\, r\right) r\, dr = 0, \qquad p \neq q,$$

where $\sqrt{\lambda_{mn}} = z_{mn}$ denotes the nth zero of $J_m(z)$. Use Mathematica to confirm this for some specific m, p, and q. (The zeros of J_m are transcendental numbers, but Mathematica's function BesselJZero returns approximations of them.)