## Green's Functions for the 2D Poisson Equation

Math 330

We will examine the Poisson equation

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

which models equilibrium phenomena (such as electrostatic or gravitational potential).

First, recall a few facts from multivariable calculus:

- The **gradient** of u(x,y) is a vector of partial derivatives:  $\nabla u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}$ .
- The **divergence** of a vector field  $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  is:  $\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ .
- The divergence theorem says

$$\iint_{\Omega} \operatorname{div} \mathbf{F} \ dA = \oint_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \ ds$$

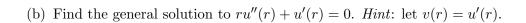
where **F** is a vector field,  $\Omega$  is a region with boundary  $\partial\Omega$ , and **n** is the outward pointing unit normal vector at each point of  $\partial\Omega$ .

- 1. Let  $f(x,y) = \delta_{\xi,\eta}$  be the 2D delta function at  $(\xi,\eta) \in \mathbb{R}^2$ , and let  $G_0(x,y;\xi,\eta)$  solve the Poisson equation for this f. Explain why  $-\Delta G = 0$  for all  $(x,y) \neq (\xi,\eta)$ .
- **2.** Explain why  $G(x, y; \xi, \eta)$  should really be a function of r alone, where  $r = \sqrt{(x \xi)^2 + (y \eta)^2}$ .
- 3. In this case, we seek a radially-symmetric solution to the 2D Laplace Equation. In polar coordinates, the Laplace equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

We want a solution  $u(r,\theta)$  that in fact depends only on r.

(a) Simplify the PDE above in the case that  $u(r, \theta) = u(r)$ .



**4.** We now have 
$$G(x, y; \xi, \eta) = a + b \ln(r)$$
, where  $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$ , and we need  $-\Delta G = \delta_{\xi, \eta}$ . Why can we choose  $a = 0$ ?

**5.** Let D be a disk of radius  $\epsilon > 0$  centered at  $(\xi, \eta)$ , and let  $C = \partial D$ . Integrate  $-\Delta G = \delta_{\xi, \eta}$  over D to solve for b.

**6.** Write the Green's function for the 2D Poisson equation.