MATH 330 • 28 Nov 2023

MOTIVATION:

Linear system of n equations and n unknowns:

$$A\vec{x} = \vec{b} \qquad \text{so} \qquad \vec{x} = A^{-1}\vec{b}$$

$n \times n$ invertible matrix

We could write $\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2 + \cdots + b_n\vec{e}_n$ where $\vec{e}_j$ is the standard basis vector for the j^{th} component of $\mathbb{R}^n$.

$b_j = \vec{b}\cdot\vec{e}_j$

Then let $\vec{x}_j$ solve $A\vec{x}_j = \vec{e}_j$, so $\vec{x}_j = A^{-1}\vec{e}_j$.

response of the system to $\vec{e}_j$

"unit impulse force" concentrated on the j^{th} component

[If you understand the "influence" of each $\vec{e}_j$, then you can solve the system for any vector $\vec{b}$.]

Can we do something similar for differential equations?

Example: $L(u(x)) = f(x)$

unknown fn

known function

L is a linear operator e.g. $\frac{d^2}{dx^2}$

Now we need a solution $u(x)$ in an infinite-dimensional function space $C^k(\mathbb{R}^n)$.

NOTE: Instead of dot products, we have inner products

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\cdot g(x)\,dx$$

integral, rather than a finite sum

We don't have basis vectors but we have the delta function

Recall. $\delta_\xi(x) = 0$ if $x \neq \xi$

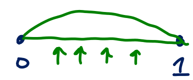
and $\displaystyle\int_{-\infty}^{\infty} \delta_\xi(x)\, f(x)\, dx = f(\xi)$

$$\boxed{\delta_\xi(x) \text{ is a "unit impulse" at } x = \xi}$$

We want to characterize the "response" to this unit impulse, then integrate the response over all points ξ.

Generalized Functions

Math 330



$L(u) = -c\frac{d^2u}{dx^2}$

Consider the boundary-value problem

$c>0$

$$-c\frac{d^2u}{dx^2} = f(x), \qquad u(0) = u(1) = 0,$$

which models the deflection of a unit-length elastic bar of stiffness c subject to a force $f(x)$.

1. Let $f(x) = \delta_\xi(x)$ for some $\delta \in (0,1)$. The differential equation is now

$$-c\frac{d^2u}{dx^2} = \delta_\xi(x).$$

Integrate twice to find $u(x)$. Your answer should contain two integration constants.

integrate once: $-c\frac{du}{dx} = \sigma_\xi(x) + a$

step function

integrate again: $-c\,u(x) = \rho_\xi(x) + ax + b$

ramp function

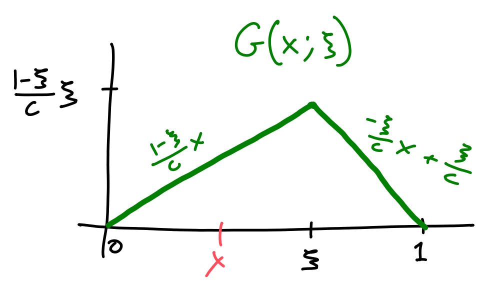
alternatively: $u(x) = \frac{-1}{c}\rho_\xi(x) + ax + b$

2. Apply the boundary conditions $u(0) = u(1) = 0$ to solve for your integration constants.

$$G_\xi(x) = G(x;\xi) = u(x) = \frac{-1}{c}\rho_\xi(x) + \frac{1-\xi}{c}x$$

the Green's function for this BVP

3. Let $G(x;\xi)$ be the solution you found in #2. Sketch the graph of $G(x;\xi)$.



for $\xi < x < 1$:

$$-\frac{1}{c}(x-\xi) + \frac{1-\xi}{c}x$$

$$= \frac{-x+\xi+x-\xi x}{c}$$

$$= \frac{\xi-\xi x}{c}$$

4. Complete the following sentences.

(a) The function $G(x;\xi)$ is called

the Green's function for this BVP

(b) We interpret $G(x;\xi)$ as

the response to a unit impulse at $x = \xi$

(c) For a general forcing function $f(x)$, the solution to the BVP is given by

$$u(x) = \int_0^1 G(x;\xi)\, f(\xi)\, d\xi$$

5. For a constant unit force $f(x) = 1$ for $0 < x < 1$, find the solution $u(x)$ to the BVP.

$$G(x;\xi) = \begin{cases} \frac{1-\xi}{c} x, & 0 \le x \le \xi \\ -\frac{\xi}{c} x + \frac{\xi}{c}, & \xi \le x \le 1 \end{cases} = \begin{cases} \frac{1-x}{c}\xi, & 0 \le \xi \le x \\ -\frac{x}{c}\xi + \frac{x}{c} & x \le \xi \le 1 \end{cases}$$

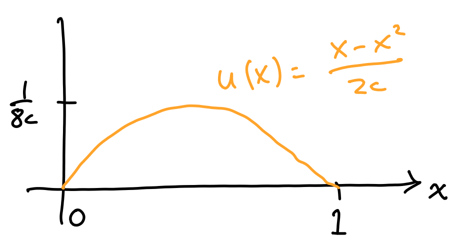
$$u(x) = \int_0^1 G(x;\xi)\, 1\, d\xi$$

$$= \underbrace{\int_0^x \frac{1-x}{c}\cdot\xi\, d\xi}_{\xi < x} + \underbrace{\int_x^1 \left(\frac{-x}{c}\xi + \frac{x}{c}\right) d\xi}_{x < \xi}$$

$$= \left[\frac{1-x}{c}\cdot\frac{1}{2}\xi^2\right]_0^x + \left[\frac{-x}{c}\cdot\frac{1}{2}\xi^2 + \frac{x}{c}\xi\right]_x^1$$

$$= \left[\frac{1-x}{c}\cdot\frac{1}{2}x^2 - 0\right] + \left[\frac{-x}{2c} + \frac{x}{c} - \left(\frac{-x^3}{2c} + \frac{x^2}{c}\right)\right]$$

$$= \frac{x^2 - \cancel{x^3}}{2c} + \frac{-x + 2x \cancel{+ x^3} - 2x^2}{2c} = \boxed{\frac{-x^2 + x}{2c} = u(x)}$$



We didn't do this in class. This example is also in the text.

Consider the boundary-value problem

$$-\frac{d^2u}{dx^2} + \omega^2 u = f(x), \qquad u(0) = u(1) = 0, \qquad \omega > 0.$$

6. First, let $f(x) = \delta_\xi(x)$ for $0 < \xi < 1$. Show that

$$G(x;\xi) = \begin{cases} a\sinh(\omega x), & x < \xi, \\ b\sinh(\omega(1-x)), & x > \xi \end{cases}$$

satisfies the BVP for $0 \le x < \xi$ and $\xi < x \le 1$.

For $0 \le x < \xi$: $\frac{d^2u}{\partial x^2} = a\omega^2 \sinh(\omega x)$,

so $-\frac{d^2u}{\partial x^2} + \omega^2 u = -a\omega^2\sinh(\omega x) + \omega^2 \cdot a\sinh(\omega x) = 0 = \delta_\xi(x)$

(since $x \neq \xi$)

For $\xi < x \le 1$: $\frac{\partial^2 u}{\partial x^2} = b\omega^2 \sinh(\omega(1-x))$,

so $-\frac{d^2u}{\partial x^2} + \omega^2 u = b\omega^2 \sinh(\omega(1-x)) + \omega^2 \cdot b\sinh(\omega(1-x)) = 0 = \delta_\xi(x)$

7. If $G(x;\xi)$ is to be continuous and $\frac{dG}{dx}(x;\xi)$ has a jump discontinuity that matches that of $\sigma_\xi(x)$, what system of equations can you solve for a and b?

Continuity at $x = \xi$ requires: $\boxed{a\cdot\sinh(\omega\xi) = b\cdot\sinh(\omega(1-\xi))}$

Also, at $x=\xi$, $u'(x)$ must have a jump discontinuity of size -1 to match the jump discontinuity of $\int \delta_\xi(x)\,dx$.

Since $\frac{\partial G}{\partial x} = \begin{cases} a\omega\cosh(\omega x), & x < \xi \\ -b\omega\cosh(\omega(1-x)), & x > \xi, \end{cases}$ we need:

solve these equations for a and b

$$\boxed{a\omega\cdot\cosh(\omega\xi) - 1 = -b\omega\cdot\cosh(\omega(1-\xi))}$$

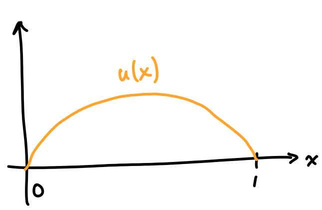
8. Solving for a and b yields

$$G(x;\xi) = \begin{cases} \frac{\sinh(\omega(1-\xi))\sinh(\omega x)}{\omega\sinh(\omega)}, & x \le \xi, \\ \frac{\sinh(\omega(1-x))\sinh(\omega\xi)}{\omega\sinh(\omega)}, & x > \xi. \end{cases}$$

Now integrate to find $u(x)$ that solves the BVP with $f(x) = 1$. Plot your solution.

$$u(x) = \int_0^1 G(x;\xi)\cdot 1\, d\xi = \int_0^x \frac{\sinh(\omega(1-x))\ \sinh(\omega\xi)}{\omega\cdot\sinh(\omega)}\,d\xi + \int_x^1 \frac{\sinh(\omega x)\cdot\sinh(\omega(1-\xi))}{\omega\sinh(\omega)}\,d\xi$$

$$= \frac{\sinh(\omega(1-x))(\cosh(\omega\xi)-1)}{\omega^2\sinh(\omega)} + \frac{\sinh(\omega x)(\cosh(\omega(1-x))-1)}{\omega^2\sinh(\omega)}$$



$$u(x) = \frac{1}{\omega^2} - \frac{\sinh(\omega x) + \sinh(\omega(1-x))}{\omega^2\sinh(\omega)} \quad \text{for } 0 \le x \le 1$$