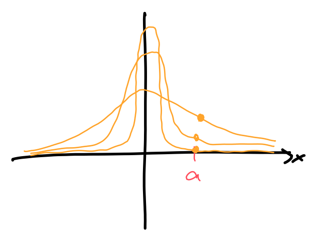
Generalized Functions

Math 330

1. Let $\boxed{g_n(x) = \dfrac{n}{\pi(1+n^2x^2)}}$ for $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

(a) Make some plots of $g_n(x)$ for various positive integers n. How does the shape of the graph depend on n?

(b) If $a \neq 0$, what is $\lim_{n\to\infty} g_n(a)$? What is $\lim_{n\to\infty} g_n(0)$?



If $a \neq 0$, $\lim_{n\to\infty} g_n(a) = 0$

$\lim_{n\to\infty} g_n(0) =$ DNE (∞)

(c) What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\,dx$? $= \lim_{n\to\infty} 1 = 1$

$$\int_{-\infty}^{\infty} g_n(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{n}{1+n^2x^2}\,dx = \frac{1}{\pi}\arctan(nx)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\lim_{x\to\infty}\arctan(nx) - \lim_{x\to-\infty}\arctan(nx)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{+\pi}{2}\right] = 1$$

(d) What function is equal to $\lim_{n\to\infty} g_n(x)$?

There is no function that is zero except at one point and yet integrates to 1.

Define $\underbrace{\delta(x)}$ to be $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

This is a "generalized function" or "distribution" called the Dirac delta function.

2. Choose your favorite continuous function $u(x)$. (Everyone at your table should choose a different function.) Explore

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)u(x)\,dx.$$

What do you observe?

Most of the time, it seems that $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\underbrace{u(x)}_{\text{"test function"}}\,dx = u(0)$

PROPERTY: $\underbrace{\int_{-\infty}^{\infty} \delta(x)\,u(x)\,dx}_{\text{This integral selects the value of } u(x) \text{ at } x=0.} = u(0)$ for all continuous functions $u(x)$.

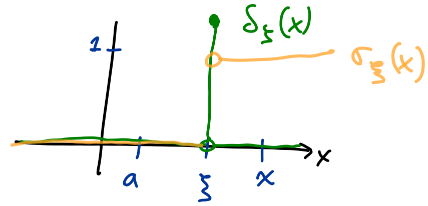
DEFINITION: $\underbrace{\delta_\xi(x) = \delta(x-\xi)}$

shifts the delta function so that it is zero for all $x \neq \xi$

$\delta(x)$ has area 1 concentrated at $x=0$

$\delta_\xi(x)$ has area 1 concentrated at $x=\xi$

assume $a < \xi < x$



3. Let $\sigma_\xi(x) = \int_a^x \delta_\xi(t)\,dt$? Sketch a graph of $\sigma_\xi(x)$.

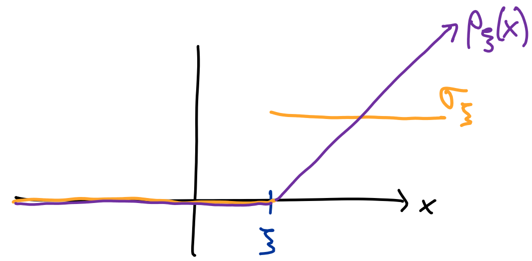
Unit step function

$$\sigma_\xi(x) = \int_a^x \delta_\xi(t)\,dt = \begin{cases} 0 & \text{if } x < \xi \\ 1 & \text{if } x > \xi \end{cases}$$

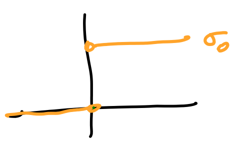
4. Let $\rho_\xi(x) = \int_a^x \sigma_\xi(t)\,dt$? Sketch a graph of $\rho_\xi(x)$.

ramp function

$$\rho_\xi(x) = \begin{cases} 0 & \text{if } x < \xi \\ x-\xi & \text{if } x > \xi \end{cases}$$



5. What is $\dfrac{d\sigma_\xi}{dx}$?



Integral of $\delta_\xi(x)$ is $\sigma_\xi(x)$.

So, derivative of $\sigma_\xi(x)$ is $\delta_\xi(x)$.

$$\frac{d\sigma_\xi}{dx} = \delta_\xi$$

6. Let $f(x) = \begin{cases} -x, & x < 1 \\ x^2, & x > 1. \end{cases}$

(a) Write $f(x)$ as the sum of a continuous function $g(x)$ plus a step function σ_ξ.

$$f(x) = g(x) + 2\sigma_1(x), \quad \text{where} \quad g(x) = \begin{cases} -x & \text{if } x < 1 \\ x^2 - 2 & \text{if } x > 1 \end{cases}$$

(b) Differentiate $f(x)$ in the context of generalized functions.

$$f'(x) = g'(x) + 2\delta_1(x)$$

7. Let $f(x) = \begin{cases} x, & -1 < x < 0 \\ x^2, & 0 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$

Write $f(x)$ as the sum of a continuous function and step functions. Then differentiate $f(x)$ in the context of generalized functions.

$$f(x) = g(x) - \sigma_{-1}(x) - 9\sigma_3(x), \quad \text{where} \quad g(x) = \begin{cases} 0, & x < -1 \\ x+1, & -1 < x < 0 \\ 1 + x^2, & 0 < x < 3 \\ 10, & x > 3 \end{cases}$$

Then: $f'(x) = g'(x) - \delta_{-1}(x) - 9\delta_3(x)$

8. Does $\delta(x)$ have a Fourier series?

(a) Find the Fourier coefficients of $\delta(x)$.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos(kx)\,dx = \frac{1}{\pi}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin(kx)\,dx = 0$$

Thus, $\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi}\left(\cos(x) + \cos(2x) + \cos(3x) + \cdots\right)$

(b) Does the Fourier series you found in part (a) converge to $\delta(x)$? (Plot some partial sums.)

The Fourier series has a spike at $x=0$, but it doesn't converge to zero for $x \neq 0$.

9. Let $s_n(x) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{n}\cos(kx)$.

(a) What is $\displaystyle\int_{-\pi}^{\pi} s_n(x)\,dx$?

$\displaystyle\int_{-\pi}^{\pi} s_n(x)\,dx = 1$ for all positive integers n.

(b) Explore $\displaystyle\lim_{n\to\infty}\int_{-\pi}^{\pi} s_n(x)u(x)\,dx$ for your choice of continuous functions $u(x)$. What do you observe?

$\displaystyle\lim_{n\to\infty}\int_{-\pi}^{\pi} s_n(x)\,u(x)\,dx = u(0)$ for all continuous functions $u(x)$

The sequence of functions $s_1(x), s_2(x), s_3(x), \ldots$ is said to "weakly converge" to $\delta(x)$.

See page 230 in the textbook for details.