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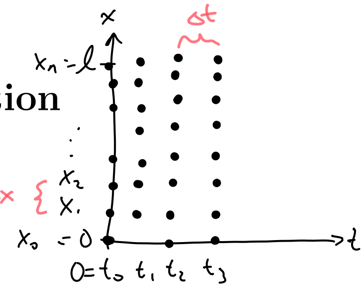
Finite Differences for the Wave Equation

Math 330



Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le \ell, \quad t \ge 0$$

with constant wave speed $c > 0$. We impose boundary conditions $u(t, 0) = u(t, \ell) = 0$ and initial conditions $u(0, x) = f(x)$ and $\frac{\partial u}{\partial t}(0, x) = g(x)$. As before, we adopt a uniformly spaced mesh $t_j = j\Delta t$ and $x_m = m\Delta x$, where $\Delta x = \frac{\ell}{n}$.

Let $u_{j,m} = u(t_j, x_m)$

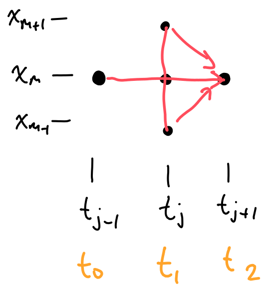
1. Convert the wave equation into a finite difference equation by replacing the second derivatives with their centered difference approximations. What is the truncation error?

$$\frac{u_{j-1,m} - 2u_{j,m} + u_{j+1,m}}{(\Delta t)^2} = c^2 \frac{u_{j,m-1} - 2u_{j,m} + u_{j,m+1}}{(\Delta x)^2}$$

error: $O(\Delta t^2)$ $\qquad$ $O(\Delta x^2)$

2. Rearrange your equation to write $u_{j+1,m}$ in terms of u at previous time steps.

Let $\sigma = \frac{c \cdot \Delta t}{\Delta x}$. Then:



$$\underbrace{u_{j+1,m}}_{\text{time index } j+1} = \underbrace{\sigma^2 u_{j,m-1} + 2(1-\sigma^2) u_{j,m} + \sigma^2 u_{j,m+1}}_{\text{time index } j} - \underbrace{u_{j-1,m}}_{\text{time index } j-1}$$

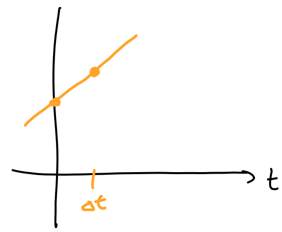
3. Write the iterative system in matrix form: $\mathbf{u}^{(j+1)} = B\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}$.

$$\begin{bmatrix} u_{j+1,1} \\ u_{j+1,2} \\ u_{j+1,3} \\ \vdots \\ u_{j+1,n-1} \end{bmatrix} = \begin{bmatrix} 2(1-\sigma^2) & \sigma^2 & 0 & 0 & \cdots & 0 \\ \sigma^2 & 2(1-\sigma^2) & \sigma^2 & 0 & & \\ 0 & \sigma^2 & 2(1-\sigma^2) & \ddots & & \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & & & & & \\ 0 & & & & & 2(1-\sigma^2) \end{bmatrix} \begin{bmatrix} u_{j,1} \\ u_{j,2} \\ u_{j,3} \\ \vdots \\ u_{j,n-1} \end{bmatrix} - \begin{bmatrix} u_{j-1,1} \\ u_{j-1,2} \\ u_{j-1,3} \\ \vdots \\ u_{j-1,n-1} \end{bmatrix}$$

4. Note that we have produced a *second order iterative scheme*, since computing each subsequent time step $\mathbf{u}^{(j+1)}$ requires knowing the values of the preceding two: $\mathbf{u}^{(j)}$ and $\mathbf{u}^{(j-1)}$.

How, therefore, can we get this method started when we only have initial values at time 0? Recall that the initial conditions for the wave equation are $u(0,x) = f(x)$ and $\frac{\partial u}{\partial t}(0,x) = g(x)$.

$$u_{1,m} = \underbrace{f(x_m)}_{\text{initial position}} + \underbrace{g(x_m)}_{\text{initial velocity}} \underbrace{\Delta t}_{\text{time interval}}$$



solve for $g(x_m) = \dfrac{u_{1,m} - f(x_m)}{\Delta t}$

$$\frac{\partial u}{\partial t}(0,x_m) = g(x_m) = \frac{u_{1,m} - u_{0,m}}{\Delta t}$$

forward difference approx. for $\frac{\partial u}{\partial t}(0,x) = g(x)$

$\boxed{\text{Error: } O(\Delta t)}$

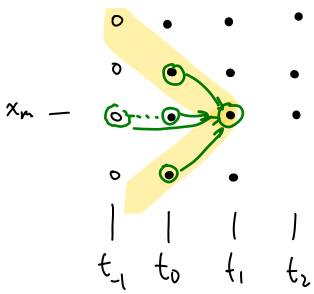
5. To avoid introducing an $O(\Delta t)$ error into our approximation scheme, we can instead proceed as follows. If we imagine a row of "phantom" grid points at time $t_{-1} = -\Delta t$, then the we can approximate the initial velocity using the centered difference approximation:

$$\underbrace{g(x_m) = \frac{\partial u}{\partial t}(0, x_m) = \frac{u_{1,m} - u_{-1,m}}{2\Delta t}} \qquad \text{error: } O(\Delta t^2) \qquad (*)$$

Using your answer to #2 above, we can write the partial difference equation at $t = 0$:

$$j=0: \qquad u_{1,m} = \sigma^2 u_{0,m-1} + 2(1-\sigma^2)u_{0,m} + \sigma^2 u_{0,m+1} - u_{-1,m} \qquad (**)$$

Write equation $(*)$ in the form $u_{-1,m} = \ldots$, then substitute this result into equation $(**)$ to eliminate $u_{-1,m}$. Then solve for $u_{1,m}$.



First: $u_{-1,m} = u_{1,m} - 2\Delta t\, g(x_m)$

Then: $u_{1,m} = \sigma^2 u_{0,m-1} + 2(1-\sigma^2)u_{0,m} + \sigma^2 u_{0,m+1} - \left(u_{1,m} - 2\Delta t\, g(x_m)\right)$

$$\boxed{u_{1,m} = \underbrace{\frac{\sigma^2}{2} u_{0,m-1} + (1-\sigma^2)u_{0,m} + \frac{\sigma^2}{2} u_{0,m+1} + \Delta t\, g(x_m)}_{\text{known quantities from initial conditions}}}$$

6. Download the Mathematica file for the wave equation from the course web site. Complete the missing portions of code and simulate solutions for your choice of wave speeds and initial conditions. What do you observe?

7. Can this numerical scheme become unstable? Perform a stability analysis to find out. Substitute $u_{j,m} = e^{ikx_m}\lambda^j$ into your numerical scheme found in #2 above. Show that

$$\lambda^2 = \left[2 - 4\sigma^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right]\lambda - 1.$$

Hint: $2\sin^2\theta = 1 - \cos(2\theta)$.

Now use the quadratic formula to find

$$\lambda = \alpha \pm \sqrt{\alpha^2 - 1}, \qquad \text{where} \quad \alpha = 1 - 2\sigma^2 \sin^2\left(\frac{k\Delta x}{2}\right).$$

Thus, there are two different magnification factors associated with each complex exponential, which is a consequence of the scheme being of second order. Stability requires that both satisfy $|\lambda| \le 1$. For what values of σ does this occur?

$$u_{j+1,m} = \sigma^2 u_{j,m-1} + 2(1-\sigma^2)u_{j,m} + \sigma^2 u_{j,m+1} - u_{j-1,m}$$

$$\cancel{e^{ikx_m}}\,\underline{\lambda^{j+1}} = \sigma^2 e^{ik(\cancel{x_m}-\Delta x)}\underline{\lambda^j} + 2(1-\sigma^2)\cancel{e^{ikx_m}}\,\underline{\lambda^j} + \sigma^2 e^{ik(\cancel{x_m}+\Delta x)}\underline{\lambda^j} - \cancel{e^{ikx_m}}\,\underline{\lambda^{j-1}}$$

$$\lambda^2 = \sigma^2 \underline{e^{-ik\Delta x}}\lambda + 2(1-\sigma^2)\lambda + \sigma^2 \underline{e^{ik\Delta x}}\lambda - 1$$

$$\left[\begin{aligned} e^{-ik\Delta x} + e^{ik\Delta x} &= \cos(-k\Delta x) + i\sin(-k\Delta x) + \cos(k\Delta x) + i\sin(k\Delta x) \\ &= \cos(k\Delta x) - i\sin(k\Delta x) + \cos(k\Delta x) + i\sin(k\Delta x) \end{aligned}\right]$$

$$\lambda^2 = \lambda\left[2\sigma^2\cos(k\Delta x) + 2 - 2\sigma^2\right] - 1$$

$$\lambda^2 = \lambda\left[2 + 2\sigma^2(\cos(k\Delta x) - 1)\right] - 1$$

$2\sin^2\theta = 1 - \cos(2\theta)$ with $\theta = \frac{1}{2}k\Delta x$.

$-2\sin^2\left(\frac{1}{2}k\Delta x\right) = \cos(k\Delta x) - 1$

$$\lambda^2 = \lambda\underbrace{\left[2 - 4\sigma^2\sin^2\left(\tfrac{1}{2}k\Delta x\right)\right]}_{\text{let this be } 2\alpha, \text{ so } \alpha = 1 - 2\sigma^2\sin^2\left(\frac{1}{2}k\Delta x\right)} - 1$$

So we have $\lambda^2 - 2\alpha\lambda + 1 = 0$, and thus $\lambda = \dfrac{2\alpha \pm \sqrt{4\alpha^2 - 4}}{2} = \alpha \pm \sqrt{\alpha^2 - 1}$ by the quadratic formula.

- If $|\alpha| < 1$, then λ is complex: $\lambda = \alpha \pm i\sqrt{1 - \alpha^2}$.
  So $|\lambda|^2 = \alpha^2 + (1 - \alpha^2) = 1$, meaning that $|\lambda| = 1$ and the scheme is stable.
- If $|\alpha| = 1$, then $\lambda = \alpha = \pm 1$ and the scheme is stable.
- If $|\alpha| > 1$, then λ is real. Since $\alpha = 1 - 2\sigma^2\sin^2\left(\frac{1}{2}k\Delta x\right) \le 1$, it suffices to consider $\alpha < -1$.
  In this case, $\alpha - \sqrt{\alpha^2 - 1} < -1$, so the scheme is unstable.

Thus, stability requires $|\alpha| < 1$, which implies $-1 \le 1 - 2\sigma^2\sin^2\left(\frac{1}{2}k\Delta x\right) \le 1$.

This requires $0 \le \sigma^2 \le 1$, which implies $0 \le \sigma = \frac{c\,\Delta t}{\Delta x} \le 1$.

Therefore, we again have stability for wave speeds satisfying $0 \le c \le \frac{\Delta x}{\Delta t}$.