Finite Differences for the Transport Equation

Math 330

Recall the transport equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(0,x) = f(x), \qquad -\infty < x < \infty.$$

1. Use forward difference approximations to convert the transport equation into a finite difference equation. Let $u_{j,m} = u(t_j, x_m)$ and write your equation in the form

$$u_{j+1,m} =$$

where the right side of the equation involves $u_{j,m}$ and $u_{j,m+1}$.

2. Write your partial difference equation as a vector equation:

$$\mathbf{u}^{(j+1)} = A\mathbf{u}^{(j)}$$

where $\mathbf{u}^{(j)}$ is the vector whose mth entry is $u_{j,m}$.

3. Download the Mathematica file for the transport equation from the course website. Complete the specification of matrix A in the code. Run the code for wave speeds $c = 0.5$, $c = -0.5$, $c = -1$, $c = -1.5$, and other values of your choice. What do you observe? How can you explain your observations?

4. Perform a stability analysis:

(a) Let $u_{j,m} = e^{ikx_m}$ and find λ such that $u_{j+1,m} = \lambda u_{j,m}$.

(b) Recalling that $|a + bi|^2 = a^2 + b^2$ for real numbers a and b, show that

$$|\lambda|^2 = 1 + 2\sigma(1 + \sigma)(1 - \cos(k\Delta x)).$$

(c) Conclude that $|\lambda| \leq 1$ if and only if $-\frac{\Delta x}{\Delta t} \leq c \leq 0$.

5. Now use a backward difference approximation for $\frac{\partial u}{\partial x}$ and a forward difference approximation for $\frac{\partial u}{\partial t}$. For what wave speeds is this approximation scheme stable?

6. Would a centered difference approximation for $\frac{\partial u}{\partial x}$ produce an approximation scheme that works for both positive and negative wave speeds? Try it and find out!