Implicit Scheme for the Heat Equation

Math 330

Consider the heat equation with homogeneous Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \qquad u(t,0) = 0, \qquad u(t,\ell) = 0, \qquad u(0,x) = f(x)$$

We will derive a partial difference equation using the backward difference in time. This results in the *implicit scheme* for approximating solutions to the heat equation.

1. Substitute a backward difference for $\frac{\partial u}{\partial t}$ and a centered difference for $\frac{\partial^2 u}{\partial x^2}$ into heat equation. Let $u_{j,m} = u(t_j, x_m)$. Write your equation in the form

$$u_{m-1,j} = \dots$$

2. Write your partial difference equation as a vector equation:

$$\mathbf{u}^{(j-1)} = A\mathbf{u}^{(j)}$$

where $\mathbf{u}^{(j)}$ is the vector whose mth entry is $u_{j,m}$.

3. Download the Mathematica file for the implicit scheme from the course website. Complete the specification of matrix A in the code. Run the code for several choices of step sizes and initial conditions. What do you observe?

4. Perform a stability analysis of the implicit scheme. That is, let $u_{j,m} = e^{ikx_m}$ and simplify the difference equation as much as possible. Find a value λ such that $u_{j,m} = \lambda u_{j-1,m}$. Under what conditions is $|\lambda| \leq 1$?

Crank-Nicolson Scheme for the Heat Equation

Math 330

Consider the explicit (forward) and implicit (backward) schemes for the heat equation:

$$\text{explicit:} \qquad \frac{u_{j+1,m} - u_{j,m}}{\Delta t} = \gamma \frac{u_{j,m+1} - 2u_{j,m} + u_{j,m-1}}{(\Delta x)^2}$$

$$\text{implicit:} \qquad \frac{u_{j+1,m} - u_{j,m}}{\Delta t} = \gamma \frac{u_{j+1,m+1} - 2u_{j+1,m} + u_{j+1,m-1}}{(\Delta x)^2}$$

1. Add the two difference equations above and obtain a new difference equation of the form

$$u_{j+1,m} - u_{j,m} = \frac{\mu}{2}\Big[\qquad\qquad ? \qquad\qquad \Big]$$

2. Collect all the terms with time index $j+1$ on the left, and all the terms with time index j on the right. Then write your equation in vector form $A\mathbf{u}^{(j+1)} = B\mathbf{u}^{(j)}$ for some matrices A and B.

3. Download the Mathematica file for the Crank-Nicolson scheme from the course website. Complete the specification of matrices A and B in the code. Run the code for several choices of step sizes and initial conditions. What do you observe?

4. Perform a stability analysis of the implicit scheme. It may help to let $u_{j,m} = e^{ikx_m}\lambda^j$ and solve for λ. Under what conditions is $|\lambda| \leq 1$?