FINITE DIFFERENCE APPROXIMATIONS

from last time: error

* Forward difference: $u'(x) \approx \dfrac{u(x+h) - u(x)}{h}$ $\quad O(h)$

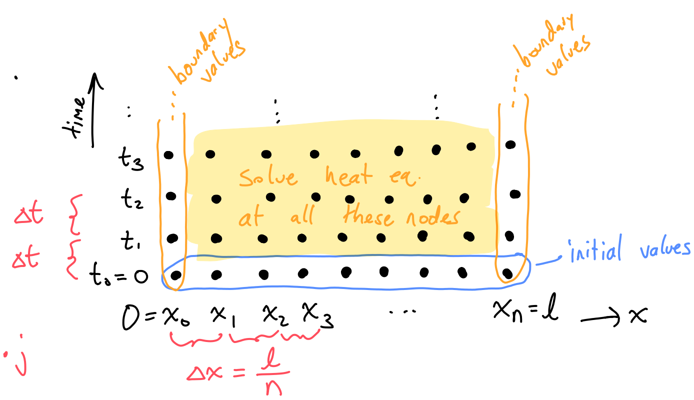
Backward difference: $u'(x) \approx \dfrac{u(x) - u(x-h)}{h}$ $\quad O(h)$

Centered difference: $u'(x) \approx \dfrac{u(x+h) - u(x-h)}{2h}$ $\quad O(h^2)$

* Centered difference for the second deriv: $u''(x) \approx \dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}$ $\quad O(h^2)$

APPROXIMATING THE HEAT EQUATION

We will approximate $u(t,x)$ on a rectangular grid of points (nodes).



Notation: let $u_{j,m} \approx u(t_j, x_m)$

j is time index

m is spatial index

$t_j = \Delta t \cdot j$

$x_m = \Delta x \cdot m$

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Forward difference in time:

$$\frac{\partial u}{\partial t}(t_j, x_m) \approx \frac{u(t_{j+1}, x_m) - u(t_j, x_m)}{\Delta t}$$

$t_j + \Delta t$

$$\approx \frac{u_{j+1,m} - u_{j,m}}{\Delta t}$$

Centered diff. second deriv. in position

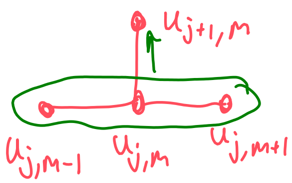
$$\frac{\partial^2 u}{\partial x^2}(t_j, x_m) \approx \frac{u(t_j, x_{m+1}) - 2u(t_j, x_m) + u(t_j, x_{m-1})}{(\Delta x)^2}$$

$$\approx \frac{u_{j,m+1} - 2u_{j,m} + u_{j,m-1}}{(\Delta x)^2}$$

Heat equation becomes: $\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$

$$\frac{u_{j+1,m} - u_{j,m}}{\Delta t} \approx \gamma \frac{u_{j,m+1} - 2u_{j,m} + u_{j,m-1}}{(\Delta x)^2}$$

$$u_{j+1,m} \approx \frac{\gamma \Delta t}{(\Delta x)^2}\left(u_{j,m+1} - 2u_{j,m} + u_{j,m-1}\right) + u_{j,m}$$



let $\mu = \frac{\gamma \Delta t}{(\Delta x)^2}$

Let $$\boxed{u_{j+1,m} = \mu\, u_{j,m+1} + (1-2\mu) u_{j,m} + \mu\, u_{j,m-1}}$$

Explicit Scheme for solving the heat equation

Let $u^{(j)}$ be the vector of approximations at interior nodes at time t_m

$$u^{(j)} = \begin{bmatrix} u_{j,1} \\ u_{j,2} \\ \vdots \\ u_{j,n-1} \end{bmatrix}$$

Then $u^{(j+1)} = A \cdot u^{(j)}$:

$$\begin{bmatrix} u_{j+1,1} \\ u_{j+1,2} \\ \vdots \\ u_{j+1,n-1} \end{bmatrix} = \begin{bmatrix} 1-2\mu & \mu & 0 & 0 \cdots 0 \\ \mu & 1-2\mu & \mu & 0 \cdots 0 \\ 0 & \mu & 1-2\mu & \mu \cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 \cdots & 0 \cdots 0 & \mu & 1-2\mu \end{bmatrix} \begin{bmatrix} u_{j,1} \\ u_{j,2} \\ \vdots \\ u_{j,n-1} \end{bmatrix}$$

← Assume zero temperature at boundary values

Stability Analysis

Math 330

We have observed that the *explicit scheme* for finding approxiate solutions to the heat equation sometimes produces solutions that exhibit crazy behavior over time. To understand this, we will examine the effect of the explicit scheme on simple functions.

Recall that the explicit scheme uses the partial difference equation:

$$u_{j+1,m} = \mu\, u_{j,m+1} + (1-2\mu)u_{j,m} + \mu\, u_{j,m-1}$$

1. Suppose that, at some time t_j, the approximate solution is given by $u(t_j, x) = e^{ikx}$, for some $k \in \mathbb{R}$. Substitute $u(t_j, x) = e^{ikx}$ into the right-hand side of the partial difference equation:

$$u_{j+1,m} = \mu e^{ikx_{m+1}} + (1-2\mu) e^{ikx_m} + \mu e^{ikx_{m-1}}$$

2. Remembering that $x_{m-1} = x_m - \Delta x$ and $x_{m+1} = x_m + \Delta x$, manipulate your equation from #1 to obtain

$$u_{j+1,m} = u_{j,m}\left[1 - 2\mu + \mu\left(e^{ik\Delta x} + e^{-ik\Delta x}\right)\right].$$

$$u_{j+1,m} = \mu e^{ik(x_m+\Delta x)} + (1-2\mu) e^{ikx_m} + \mu e^{ik(x_m-\Delta x)}$$

$$= e^{ikx_m}\left[\mu e^{ik\Delta x} + 1 - 2\mu + \mu e^{-ik\Delta x}\right]$$

$$= u_{j,m}\left[1 - 2\mu + \mu\left(e^{ik\Delta x} + e^{-ik\Delta x}\right)\right]$$

3. Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to rewrite your equation as

$$u_{j+1,m} = u_{j,m}\left[1 - 2\mu\left(1 - \cos(k\Delta x)\right)\right].$$

First: $e^{ik\Delta x} + e^{-ik\Delta x} = \cos(k\Delta x) + \cancel{i\sin(k\Delta x)} + \cos(-k\Delta x) + \cancel{i\sin(-k\Delta x)}$

$$= 2\cos(ik\Delta x)$$

Then: $u_{j+1,m} = u_{j,m}\left[1 - 2\mu + \mu\left(2\cos(k\Delta x)\right)\right]$

$$= u_{j,m}\underbrace{\boxed{\left[1 - 2\mu\left(1 - \cos(k\Delta x)\right)\right]}}_{\lambda}$$

4. Let $\lambda = 1 - 2\mu\,(1 - \cos(k\Delta x))$. The partial difference equation then becomes

$$u_{j+1,m} = \lambda u_{j,m}. \leftarrow \text{for all time indexes } j$$

What happens to our approximation if $|\lambda| < 1$? What happens if $|\lambda| > 1$?

If $|\lambda| < 1$, then the approximation converges to zero over time.

If $|\lambda| > 1$, then the approximation diverges — in fact, it grows exponentially over time.

5. Explain why $\lambda \leq 1$.

First, $1 \geq \cos(k\,\Delta x)$, so $1 - \cos(k\,\Delta x) \geq 0$. Since $\mu > 0$, this implies $-2\mu\,(1 - \cos(k\,\Delta x)) \leq 0$.

Then $\lambda = 1 - 2\mu\,(1 - \cos(k\,\Delta x)) \leq 1$.

6. Show that $-1 < \lambda$ is equivalent to $\dfrac{1}{1 - \cos(k\Delta x)} > \mu$. You may assume $\cos(k\Delta x) \neq 1$.

The following are equivalent:

$$-1 < \lambda$$

$$-1 < 1 - 2\mu(1 - \cos(k\,\Delta x))$$

$$-2 < -2\,\mu\,(1 - \cos(k\,\Delta x))$$

$$1 > \mu\,(1 - \cos(k\,\Delta x))$$

$$\frac{1}{1 - \cos(k\,\Delta x)} > \mu$$

7. Explain why $-1 < \lambda$ if $\mu < \frac{1}{2}$.

Since $1 - \cos(k\,\Delta x) \leq 2$, we have $\dfrac{1}{1 - \cos(k\,\Delta x)} \geq \dfrac{1}{2}$.

If $\mu < \frac{1}{2}$, then $\dfrac{1}{1 - \cos(k\,\Delta x)} \geq \dfrac{1}{2} > \mu$, and thus the previous problem implies $-1 < \lambda$.

8. Why might the solution be badly behaved if $\mu > \frac{1}{2}$?

If $\mu > \frac{1}{2}$, then λ will be less than -1 for some values of k and Δx. Thus the solution will diverge.

9. If we require that $\mu < \frac{1}{2}$, what restrictions does this impose on Δx and Δt?

$$\mu = \frac{\gamma\,\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \quad \text{implies that} \quad \Delta t \leq \frac{(\Delta x)^2}{2\gamma}$$

If Δx is small, then Δt must be very small!