Laplace's Equation in a Rectangle

Math 330

1. First, we should get a little more comfortable with hyperbolic sine and cosine, denoted $\sinh x$ and $\cosh x$ (hyperbolic sine is pronounced "synch"). They are defined

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

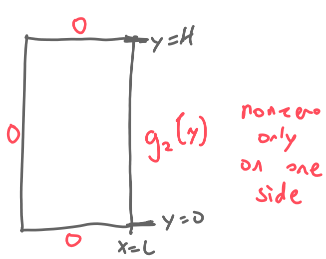
(a) Compute $\cosh 0$ and $\sinh 0$.

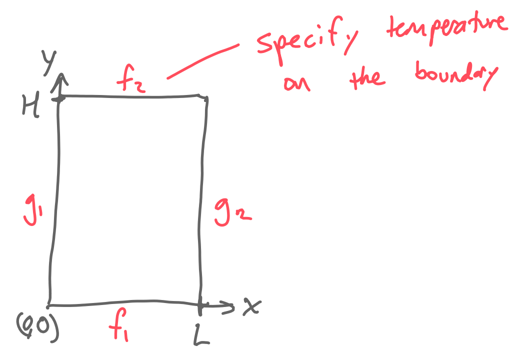
$$\{e^x, e^{-x}\}$$
$$\Updownarrow$$
$$\{\cosh(x), \sinh(x)\}$$

(b) Compute $\frac{d}{dx}\cosh x$ and $\frac{d}{dx}\sinh x$.

(c) Compute $\cosh^2 x - \sinh^2 x$.

(d) Write e^x and e^{-x} in terms of $\cosh x$ and $\sinh x$. This shows that linear combinations of e^x and e^{-x} can alternatively be written as linear combinations of $\cosh x$ and $\sinh x$.





Now let's discuss Laplace's equation,

$u(x,y)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

on the rectangular domain $D = \{(x,y) \mid 0 < x < L, 0 < y < H\}$. The solution $u(x,y)$ gives the equilibrium temperature distribution on the rectangle with prescribed boundary conditions. To keep things simple, we will assume inhomogeneous boundary conditions on only one side and homogeneous boundary conditions on the other three sides:

$$u(x,0) = 0 \qquad u(0,y) = 0 \quad \text{— } v(0)\,w(y) = 0$$
$$u(x,H) = 0 \qquad g_2(y) = u(L,y) = 10$$

$w(H) = 0$

$v(0) = 0$

$u(x,0) = v(x)\,\underbrace{w(0) = 0}$

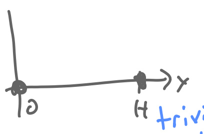
$w(0) = 0$

2. Let $u(x, y) = v(x)w(y)$. Plug this into the PDE, separate variables, set the resulting expressions equal to a constant λ, and arrive at two ODEs. What are the appropriate boundary conditions for each ODE?

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda$$

So: $v''(x) = \lambda v(x)$, $v(0)=0$ and $\boxed{w''(y) = -\lambda w(y), \quad w(0) = w(H) = 0}$

3. One of your ODEs should have two boundary conditions. *This is your eigenvalue problem.* Find the eigenvalues and associated eigenfunctions.



$\lambda < 0$: $w(y) = c_1 \cosh(\sqrt{-\lambda}\, y) + c_2 \sinh(\sqrt{-\lambda}\, y)$

$w(0) = 0 = c_1(1) + c_2(0)$

$0 = c_1$ so $w(y) = c_2 \sinh(\sqrt{-\lambda}\, y)$

$w(H) = 0 = c_2 \sinh(H\sqrt{-\lambda})$

$0 = c_2$

trivial solution $w(y)=0$

$\lambda = 0$: $w(y) = ay + b$

$w(0) = w(H)$ imply $a = b = 0$

trivial solution $w(y) = 0$

$\lambda > 0$: $w(y) = c_1 \cos(\sqrt{\lambda}\, y) + c_2 \sin(\sqrt{\lambda}\, y)$

$w(0) = 0 \Rightarrow c_1 = 0$

$w(H) = 0 \Rightarrow 0 = c_2 \sin(H\sqrt{\lambda})$

$H\sqrt{\lambda} = n\pi, \quad n \in \mathbb{Z}^+$

$\sqrt{\lambda} = \frac{n\pi}{H}$

$\lambda_n = \left(\frac{n\pi}{H}\right)^2$ eigenvalues

$\sin\left(\frac{n\pi}{H} y\right)$ eigenfunctions

4. Use the eigenvalues you found in the previous problem to solve the second ODE, with the single boundary condition. *Hint*: Express the general solution for this ODE in terms of sinh and cosh, then apply the boundary condition.

$v''(x) = \lambda v(x)$

$v(0) = 0$

Know: $\lambda > 0$

$\lambda = \left(\frac{n\pi}{H}\right)^2$

So: $v(x) = c_1 \cosh(\sqrt{\lambda}\, x) + c_2 \sinh(\sqrt{\lambda}\, x)$

$v(0) = 0 = c_1(1) + c_2(0)$

$0 = c_1$

$v(x) = c_2 \sinh(\sqrt{\lambda}\, x) = c_2 \sinh\left(\frac{n\pi}{H} x\right)$

5. You should now have solutions to $v(x)$ and $w(y)$, dependent on some set of eigenvalues (indexed by n). Confirm that you get

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{\cancel{L}\,H} x\right) \sin\left(\frac{n\pi}{\cancel{L}\,H} y\right)$$

when you take an infinite sum of these product solutions.

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{H} x\right) \sin\left(\frac{n\pi}{H} y\right)$$

6. Now use orthogonality and the boundary condition $u(L, y) = 10$ to compute the c_n coefficients. This completes the solution for $u(x, y)$. Plot your solution to a sufficient number of terms to check that it looks reasonable.

$$u(L,y) = 10 = \sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi}{H} \cdot L\right)}_{b_n} \sin\left(\frac{n\pi}{H} y\right)$$

$$b_n = \frac{2}{H} \int_0^H 10 \sin\left(\frac{n\pi}{H} y\right) dy = \frac{1-(-1)^n}{n\pi} 20$$

So:

$$c_n = \frac{20\left(1 - (-1)^n\right)}{n\pi \sinh\left(\frac{n\pi L}{H}\right)}$$

$$c_n \cdot \sinh\left(\frac{n\pi}{H} L\right) = 20 \frac{1-(-1)^n}{n\pi}$$

Series solution:

$$u(x,y) = \sum_{n=1}^{\infty} \underbrace{\frac{20\left(1-(-1)^n\right)}{n\pi \sinh\left(\frac{n\pi L}{H}\right)}}_{C_n} \sinh\left(\frac{n\pi}{H}x\right) \sin\left(\frac{n\pi}{H}y\right)$$

Laplace's Equation in a Disk

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Now we can solve Laplace's Equation on a unit disk. Note that a circular disk of radius 1 is the region $0 \leq r \leq 1$, $-\pi \leq \theta \leq \pi$ in polar coordinates. Solving Laplace's equation with an initial condition on the boundary of the disk then amounts to solving the following problem:

PDE:	$\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ ← see page 161 in our textbook
Boundary:	$u(1,\theta) = f(\theta)$
Periodicity:	$u(r,\theta+2\pi) = u(r,\theta)$
Boundedness:	$\lvert u(0,\theta)\rvert < \infty$

(The last condition above is necessary to eliminate solutions that are unbounded near the origin.)

1. Suppose that solutions have the form $u(r,\theta) = v(r)w(\theta)$. Plug this solution into the PDE and separate variables to obtain two ODEs.

plug in $u(r,\theta) = v(r)w(\theta)$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0$$

multiply both sides by: $\dfrac{r^2}{v(r)\,w(\theta)}$

$$\left[v''(r)\,w(\theta) + \frac{1}{r}v'(r)\,w(\theta) + \frac{1}{r^2}v(r)\,w''(\theta) = 0\right]\frac{r^2}{v(r)\,w(\theta)}$$

$$\frac{r^2 v''(r)}{v(r)} + \frac{r\,v'(r)}{v(r)} + \frac{w''(\theta)}{r^2 w(\theta)} = 0$$

$$\frac{r^2 v''(r) + r\,v'(r)}{v(r)} = \frac{-w''(\theta)}{w(\theta)} = \lambda$$

← Since we have a function of r alone equal to a function of θ alone

ODEs: $r^2 v''(r) + r\,v'(r) = \lambda\, v(r)$ and $w''(\theta) = -\lambda\, w(\theta)$

2. The θ-dependent problem is very much like the circular ring problem that we solved before. Write down the eigenvalues and eigenfunctions.

→ pages 69-70 in our text

$w''(\theta) = -\lambda\, w(\theta)$ with $w(\theta + 2\pi) = w(\theta)$

eigenvalues	eigenfunctions
$\lambda = 0$	constant
$\lambda = n^2,\ n \in \mathbb{Z}^+$	$\sin(n\theta)$, $\cos(n\theta)$

3. The r-dependent problem is a Cauchy-Euler equation. Look for solutions of the form $v(r) = r^k$.

$$r^2 v''(r) + r v'(r) - n^2 v(r) = 0$$

try $\underbrace{v(r) = r^k}$: $r^2 k(k-1) r^{k-2} + r k r^{k-1} - n^2 r^k = 0$

so $v'(r) = k r^{k-1}$

and $v''(r) = k(k-1) r^{k-2}$

Factor: $r^k \left(k(k-1) + k - n^2\right) = 0$

$r^k \left(k^2 - n^2\right) = 0$

Assume $r \neq 0$. Then $k^2 - n^2 = 0$, so $k = \pm n$.

So if $n \neq 0$, we have solutions $v_1(r) = r^n$ and $v_2(r) = r^{-n}$.

If $n = 0$, then the ODE is really $r^2 v''(r) + r v'(r) = 0$.

ONE SOLUTION:

$v_1(r) = \text{constant}$

ANOTHER SOLUTION:

Let $y(r) = v'(r)$, then the ODE is $r^2 y' = -ry$, or $\frac{y'}{y} = \frac{-1}{r}$.

Write this as $\frac{dy}{dr} \cdot \frac{1}{y} = \frac{-1}{r}$, and separate variables: $\int \frac{dy}{y} = \int \frac{-1}{r}\, dr$.

Integrating both sides, we have $\ln|y| = -\ln|r| + C$, or $y = \frac{c}{r}$.

Thus $v_2(r) = \int y(r)\, dr = c_1 \ln|r| + c_2$.

4. What are all product solutions $u(r, \theta) = v(r)w(\theta)$? Which ones satisfy the boundedness condition?

$n = 0$: $u(r,\theta) = c$ or $u(r,\theta) = c \cdot \ln|r|$

$n \in \mathbb{Z}^+$: $u(r,\theta) = r^n \cos(n\theta)$, $u(r,\theta) = r^n \sin(n\theta)$ or $u(r,\theta) = r^{-n} \cos(n\theta)$, $u(r,\theta) = r^{-n} \cos(n\theta)$

← These solutions have singularities as $r \to 0$, so they don't satisfy the boundedness condition.

5. Use the principle of superposition to obtain the series solution.

$$\boxed{u(r,\theta) = c + \sum_{n=1}^{\infty} \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right)}$$

Set $r = 1$: $f(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$

Now we can use the boundary condition $f(\theta)$ to solve for the Fourier coefficients a_n and b_n.