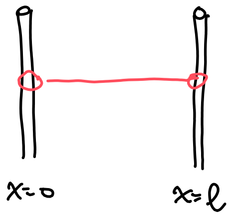
Resuming: Solving the wave equation via separation of variables

4. Rather than fixing the ends of the string, suppose we loop the ends around two frictionless rods which allow the ends to move up and down without losing energy. Now the system is modeled by



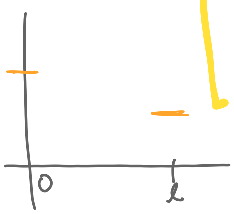
PDE:	$\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$ Wave eq.
Boundary Conditions:	$\boxed{\dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,\ell) = 0}$
Initial position:	$u(0,x) = f(x)$
Initial velocity:	$\dfrac{\partial u}{\partial t}(0,x) = g(x)$

Use the method of separation of variables to find the series solution to the wave equation with these boundary conditions.

suppose $u(t,x) = w(t)\,v(x) \longrightarrow \dfrac{\partial u}{\partial x}(t,x) = w(t)\,v'(x)$

$\dfrac{\partial u}{\partial x}(t,0) = w(t)v'(0) = 0 \Downarrow v'(0) = 0$

ODEs: BVP $\boxed{\begin{array}{l} v''(x) = \dfrac{\lambda}{c^2}\, v(x) \\ v'(0) = v'(\ell) = 0 \end{array}}$ and $w''(t) = \lambda\, w(t)$



$\lambda = 0$: $v(x) = \text{const}$

$\lambda_n < 0$: $v_n(x) = \cos\left(\frac{n\pi}{\ell} x\right)$ for $n \in \{1,2,3,\dots\}$

~~$\lambda > 0$: $v(x) = c_1 e^{\frac{\sqrt{\lambda}\, x}{c}} + c_2 e^{-\frac{\sqrt{\lambda}\, x}{c}}$~~

~~no nontrivial solutions satisfy boundary conditions~~

$\lambda = 0$: $w''(t) = 0$

so $w(t) = at + b$

$\lambda_n = -\left(\frac{n\pi c}{\ell}\right)^2 < 0$

$w_n(t) = a_n \cos\left(\sqrt{-\lambda}\, t\right) + b_n \sin\left(\sqrt{-\lambda}\, t\right)$

$w_n(t) = a_n \cos\left(\frac{n\pi c}{\ell} t\right) + b_n \sin\left(\frac{n\pi c}{\ell} t\right)$

diff. twice:

$-\left(\frac{n\pi}{\ell}\right)^2 = \frac{\lambda}{c^2}$

$-\left(\frac{n\pi c}{\ell}\right)^2 = \lambda$

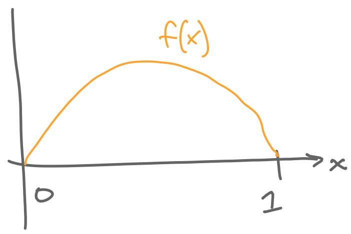
General Solution:

$$u(t,x) = \underbrace{at + b}_{\lambda = 0} + \underbrace{\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{\ell} x\right)\left[a_n \cos\left(\frac{n\pi c}{\ell} t\right) + b_n \sin\left(\frac{n\pi c}{\ell} t\right)\right]}_{\lambda < 0}$$

$$u(t,x) = a_0 t + b_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{\ell}x\right)\left[a_n \cos\left(\frac{n\pi c}{\ell}t\right) + b_n \sin\left(\frac{n\pi c}{\ell}t\right)\right]$$

5. Suppose $\ell = 1$, $c = 1$, $f(x) = x(1-x)$, and $g(x) = 0$. Solve for the coefficients and plot the solution for several values of t.

pos $u(0,x)$ vel $\frac{\partial u}{\partial t}(0,x)$



Set $t=0$: $u(0,x) = 0 + b + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{1}x\right)\left[a_n \overbrace{\cos(0)}^{1} + \cancel{b_n \sin(0)}^{0}\right]$

$$\underbrace{f(x) = b + \sum_{n=1}^{\infty} a_n \cos(n\pi x)}$$

$$b = \int_0^1 f(x)\,dx = \frac{1}{6}$$

$$\boxed{u(t,x) = \frac{1}{6} + \sum_{n=1}^{\infty} \cos(n\pi x)\left(\frac{-2(1+(-1)^n)}{(n\pi)^2}\right)\cos(n\pi c t)}$$

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx = -2\,\frac{1+(-1)^n}{(n\pi)^2}$$

Diff: $\frac{\partial u}{\partial t} = a_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{\ell}x\right)\left[a_n \cdot \frac{n\pi c}{\ell} \cdot -\underbrace{\sin\left(\frac{n\pi c}{\ell}t\right)}_{0} + b_n \cdot \frac{n\pi c}{\ell} \cdot \underbrace{\cos\left(\frac{n\pi c}{\ell}t\right)}_{1}\right]$

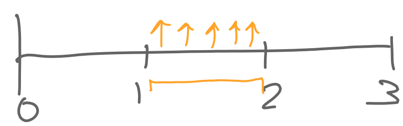
set $t=0$: $0 = a_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{1}x\right) \cdot b_n \cdot \frac{n\pi c}{1}$

so $a_0 = 0$ and $b_n = 0$ for all n.

6. Suppose $\ell = 3$, $c = 1$, $f(x) = 0$, and $g(x) = \begin{cases} 1, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$. Solve for the coefficients and plot the solution for several values of t.

no initial displacement

initial velocity



Here are the details we skipped in class:

General Solution: $u(t,x) = at + b + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}x\right)\left[a_n \cos\left(\frac{n\pi}{3}t\right) + b_n \sin\left(\frac{n\pi}{3}t\right)\right]$

with $\ell = 3$, $c = 1$

Set $t=0$: $\underbrace{u(0,x) = 0 = b + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}x\right) \cdot a_n}$

This Fourier series implies $b = 0$ and $a_n = 0$ for $n = 1, 2, 3, \ldots$

Differentiate: $\frac{\partial u}{\partial t}(t,x) = a + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}x\right)\left[-a_n \frac{n\pi}{3}\sin\left(\frac{n\pi}{3}t\right) + b_n \frac{n\pi}{3}\cos\left(\frac{n\pi}{3}t\right)\right]$

set $t=0$: $\frac{\partial u}{\partial t}(0,x) = g(x) = a + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}x\right)\left[0 + \underbrace{b_n \cdot \frac{n\pi}{3}} \cdot 1\right]$

let $c_n = b_n \cdot \frac{n\pi}{3}$

We obtain a Fourier cosine series for $g(x)$:

$$g(x) = a + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{3}x\right)$$

Solve for the coeffizients:

$$a = \frac{1}{3}\int_0^3 g(x)\,dx = \frac{1}{3}$$

$$c_n = \frac{2}{3}\int_0^3 g(x)\cos\left(\frac{n\pi}{3}x\right)dx = \frac{2\left(\sin\left(\frac{2\pi n}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right)}{n\pi}$$

$$\text{so} \quad b_n = \frac{3}{n\pi}c_n = \frac{6\left(\sin\left(\frac{2\pi n}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right)}{n^2\pi^2}$$

Thus we have the particular solution:

$$\boxed{u(t,x) = \frac{1}{3}t + \sum_{n=1}^{\infty} \frac{6\left(\sin\left(\frac{2\pi n}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right)}{n^2\pi^2}\cos\left(\frac{n\pi}{3}x\right)\sin\left(\frac{n\pi}{3}t\right)}$$

Laplace's Equation in a Rectangle

Math 330

1. First, we should get a little more comfortable with hyperbolic sine and cosine, denoted $\sinh x$ and $\cosh x$ (hyperbolic sine is pronounced "synch"). They are defined

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Compute $\cosh 0$ and $\sinh 0$.

$\cosh(0) = 1$ $\qquad$ $\sinh(0) = 0$

(b) Compute $\frac{d}{dx}\cosh x$ and $\frac{d}{dx}\sinh x$.

$\frac{d}{dx}\cosh(x) = \sinh(x)$ $\qquad$ $\frac{d}{dx}\sinh(x) = \cosh(x)$

(c) Compute $\cosh^2 x - \sinh^2 x$.

$\cosh^2(x) - \sinh^2(x) = 1$

(d) Write e^x and e^{-x} in terms of $\cosh x$ and $\sinh x$. This shows that linear combinations of e^x and e^{-x} can alternatively be written as linear combinations of $\cosh x$ and $\sinh x$.

$e^x = \cosh(x) + \sinh(x)$

$e^{-x} = \cosh(x) - \sinh(x)$

$\underbrace{v''(x) = v(x)}$ has solutions

$v(x) = c_1 e^x + c_2 e^{-x}$

or equivalently,

$v(x) = b_1 \cdot \cosh(x) + b_2 \sinh(x)$

We will continue this next week.

Now let's discuss Laplace's equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

on the rectangular domain $D = \{(x, y) \mid 0 < x < L, 0 < y < H\}$. The solution $u(x, y)$ gives the equilibrium temperature distribution on the rectangle with prescribed boundary conditions. To keep things simple, we will assume inhomogeneous boundary conditions on only one side and homogeneous boundary conditions on the other three sides:

$$u(x, 0) = 0 \qquad\qquad u(0, y) = 0$$
$$u(x, H) = 0 \qquad\qquad u(L, y) = 10$$

2. Let $u(x, y) = v(x)w(y)$. Plug this into the PDE, separate variables, set the resulting expressions equal to a constant λ, and arrive at two ODEs. What are the appropriate boundary conditions for each ODE?

3. One of your ODEs should have two boundary conditions. *This is your eigenvalue problem.* Find the eigenvalues and associated eigenfunctions.

4. Use the eigenvalues you found in the previous problem to solve the second ODE, with the single boundary condition. *Hint*: Express the general solution for this ODE in terms of sinh and cosh, then apply the boundary condition.

5. You should now have solutions to $v(x)$ and $w(y)$, dependent on some set of eigenvalues (indexed by n). Confirm that you get

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right)$$

when you take an infinite sum of these product solutions.

6. Now use orthogonality and the boundary condition $u(L, y) = 10$ to compute the c_n coefficients. This completes the solution for $u(x, y)$. Plot your solution to a sufficient number of terms to check that it looks reasonable.