Wave Equation – Separation of Variables

Math 330

1. The motion of a string held taught with fixed endpoints is modeled by:

PDE:	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
Boundary Conditions:	$u(t,0) = u(t,\ell) = 0$
Initial position:	$u(0,x) = f(x)$
Initial velocity:	$\dfrac{\partial u}{\partial t}(0,x) = g(x)$

(a) Use separation of variables to solve the wave equation. Assume that $u(t,x) = w(t)v(x)$ and produce two ordinary differential equations.

(b) Which of your ODEs gives a boundary-value problem? What are the associated eigenvalues and eigenfunctions?

(c) With the eigenvalues in hand, solve the other ODE.

(d) Apply the principle of superposition to write the series solution to the wave equation with the given boundary conditions.

Suppose $u(t,x) = w(t)\,v(x)$. Then $\frac{\partial^2 u}{\partial t^2} = w''(t)v(x)$ and $\frac{\partial^2 u}{\partial x^2} = w(t)v''(x)$.

So $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ becomes $w''(t)v(x) = c^2\, w(t)\, v''(x)$.

Separate variables to obtain $\underbrace{\dfrac{w''(t)}{w(t)} = c^2 \dfrac{v''(x)}{v(x)}}_{\text{This must be constant since the left side depends only on } t \text{ while the right side depends only on } x} = \lambda$

We thus obtain two ODEs:

$w''(t) = \lambda\, w(t)$ and $\underbrace{v''(x) = \frac{\lambda}{c^2} v(x) \quad \text{with } v(0) = v(\ell) = 0}_{\text{BVP (boundary value problem)}}$

- If $\lambda > 0$, then $v(x) = c_1 e^{\frac{\sqrt{\lambda}}{c}x} + c_2 e^{-\frac{\sqrt{\lambda}}{c}x}$

  BC $\Rightarrow$ only the trivial solution $v(x) = 0$

- If $\lambda = 0$, then $v(x) = ax + b$

  BC $\Rightarrow$ only the trivial solution

- If $\lambda < 0$, then $v(x) = c_1 \cos\left(\frac{\sqrt{-\lambda}}{c}x\right) + c_2 \sin\left(\frac{\sqrt{-\lambda}}{c}x\right)$

  BC: $v(0) = 0 \Rightarrow c_1 = 0$

  So $v(x) = c_2 \sin\left(\frac{\sqrt{-\lambda}}{c}x\right)$

  $v(\ell) = 0 \Rightarrow 0 = c_2 \sin\left(\frac{\sqrt{-\lambda}}{c}\ell\right)$

  which means $\frac{\sqrt{-\lambda}}{c}\ell = n\pi$ for $n \in \mathbb{Z}^+$

  $\sqrt{-\lambda} = \frac{n\pi c}{\ell}$

The other ODE:

$$w''(t) = \lambda\, w(t) \quad \text{becomes}$$

$$w''(t) = -\left(\frac{n\pi c}{\ell}\right)^2 w(t)$$

Eigenvalues: $\lambda_n = -\left(\frac{n\pi c}{\ell}\right)^2$

Eigenfunctions: $v_n(x) = \sin\left(\frac{n\pi}{\ell}x\right)$

This has solutions:

$$w_n(t) = d_1 \cos\left(\frac{n\pi c}{\ell}t\right) + d_2 \sin\left(\frac{n\pi c}{\ell}t\right)$$

The general solution to $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with $u(t,0) = u(t,\ell) = 0$ is:

$$u(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{\ell}x\right)\left[a_n \cos\left(\frac{n\pi c}{\ell}t\right) + b_n \sin\left(\frac{n\pi c}{\ell}t\right)\right]$$

INITIAL CONDITIONS:

$$u(0,x) = f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \end{cases}$$

$$\frac{\partial u}{\partial t}(0,x) = g(x) = 0$$

$$u(0,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{\ell}x\right)\left[a_n \overset{1}{\cos(0)} + b_n \sin(0)\right] = f(x)$$

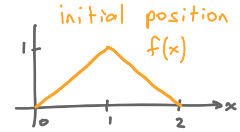
$$\underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x)}_{\text{Fourier sine series}}$$

From Section 3.4: $\frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}x\right)dx$

$$a_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right)dx$$

$$\frac{\partial u}{\partial t}(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left[-a_n \frac{n\pi c}{\ell} \overset{0}{\sin\left(\frac{n\pi c}{\ell}t\right)} + b_n \frac{n\pi c}{\ell} \overset{1}{\cos\left(\frac{n\pi c}{\ell}t\right)}\right]$$

$$\frac{\partial u}{\partial t}(0,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left[0 + b_n \frac{n\pi c}{\ell} \cos\left(\frac{n\pi c}{\ell}t\right)\right] = 0$$



$u(t,0)$

2. Suppose $\underline{\ell = 2}$, $\underline{c = 1}$, $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases}$, and $g(x) = 0$. Solve for the coefficients and plot the solution for several values of t.

zero initial velocity

Set $t=0$: $u(0,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left[a_n \underbrace{\cos(0)}_{1} + b_n \underbrace{\sin(0)}_{0}\right]$

so $\underbrace{f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}x\right)}$

Fourier sine series, so $a_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$ for $n \in \mathbb{Z}^+$

Differentiate and set $t=0$:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left[-a_n \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}t\right) + b_n \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}t\right)\right]$$

$$\frac{\partial^2 u}{\partial t^2}(0,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left[\quad 0 \quad + b_n \cdot \frac{n\pi}{2} \cdot 1\right]$$

$\frac{\partial^2 u}{\partial t^2}(0,x) = g(x)$

$$0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right) \cdot b_n \cdot \frac{n\pi}{2}$$

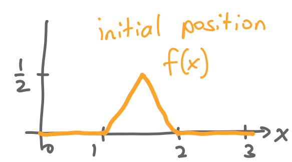
$\leftarrow$ Fourier sine series for $g(x) = 0$, so $b_n = 0$ for all n.

$$\boxed{\text{Solution: } u(t,x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}t\right)}$$

$\leftarrow$ See animation in Mathematica notebook

3. Suppose $\ell = 3$, $c = 1$,

$$f(x) = \begin{cases} x - 1, & 1 < x < \frac{3}{2} \\ 2 - x, & \frac{3}{2} \leq x < 2\,, \\ 0, & \text{otherwise} \end{cases}$$



and $g(x) = 0$. Solve for the coefficients and plot the solution for several values of t.

Set $t=0$: $u(0,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}x\right)\left[a_n \cos(0) + b_n \sin(0)\right]$

$$\underbrace{f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{3}x\right)}$$

Fourier sine series, so $a_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi}{3}x\right)dx = \frac{-6}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{3}\right) - 2\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{2n\pi}{3}\right)\right)$

As in #2, $b_n = 0$ for all n.

$$\boxed{\begin{array}{l} \text{Solution: } u(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{3}x\right)\cos\left(\frac{n\pi}{3}t\right) \\ \text{with } a_n = \frac{-6}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{3}\right) - 2\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{2n\pi}{3}\right)\right) \end{array}}$$

$\leftarrow$ See animation in Mathematica notebook

We will return to this next time:

4. Rather than fixing the ends of the string, suppose we loop the ends around two frictionless rods which allow the ends to move up and down without losing energy. Now the system is modeled by

PDE:	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
Boundary Conditions:	$\dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,\ell) = 0$
Initial position:	$u(0,x) = f(x)$
Initial velocity:	$\dfrac{\partial u}{\partial t}(0,x) = g(x)$

Use the method of separation of variables to find the series solution to the wave equation with these boundary conditions.

5. Suppose $\ell = 1$, $c = 1$, $f(x) = x(1-x)$, and $g(x) = 0$. Solve for the coefficients and plot the solution for several values of t.

6. Suppose $\ell = 3$, $c = 1$, $f(x) = 0$, and $g(x) = \begin{cases} 1, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$. Solve for the coefficients and plot the solution for several values of t.