Fourier Coefficients

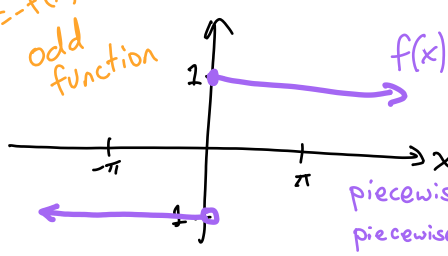
Math 330

1. Find the Fourier series of $f(x) = |x|$. Then plot $f(x)$ together with partial sums of its Fourier series.

We did this last time.

2. Find the Fourier series of $f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$. Then plot $f(x)$ together with partial sums of its Fourier series.

$f(-x) = -f(x)$ odd function



piecewise continuous

piecewise C^1

piecewise C^n for any pos. integer n

Fourier coefficients:

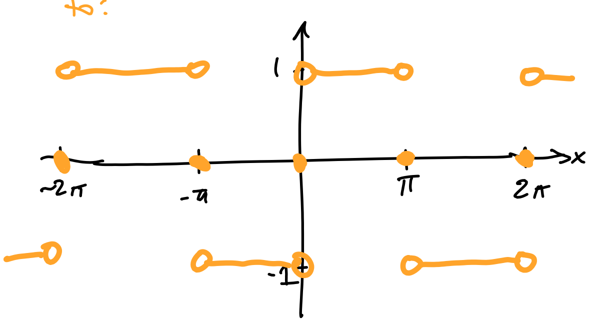
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{f(x)}\ \underset{\text{even}}{\cos(kx)}}_{\text{odd}}\,dx = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{f(x)}\ \underset{\text{odd}}{\sin(kx)}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(kx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{-1}{k}\cos(kx)\right]_0^{\pi} = \frac{-2}{k\pi}\left[\cos(k\pi) - \cos(0)\right]$$

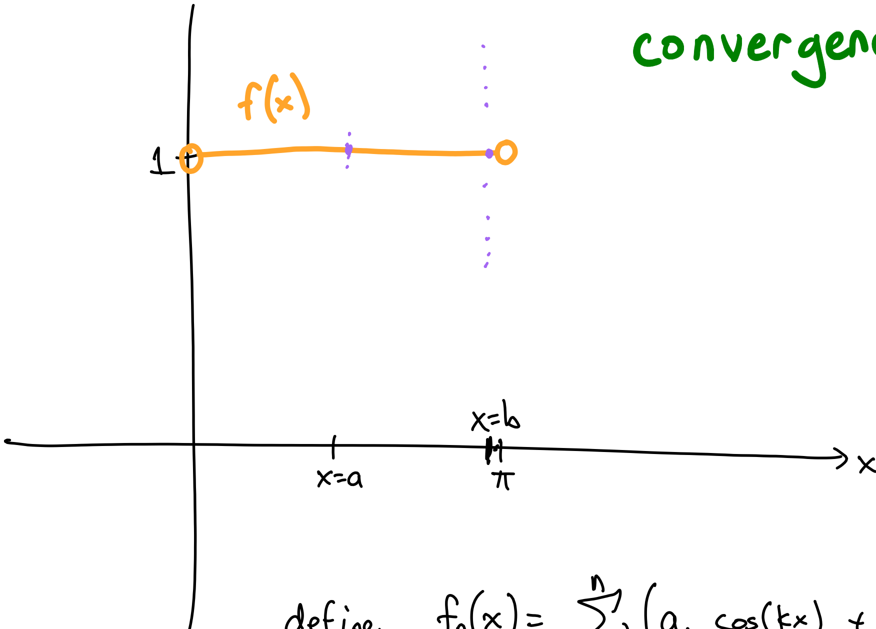
$$= \frac{-2}{k\pi}\left((-1)^k - 1\right) = \frac{1-(-1)^k}{k\pi}\cdot 2$$

Fourier series for $f(x)$ converges to:



$$\boxed{\text{FOURIER SERIES: } f(x) \sim \sum_{k=1}^{\infty} \frac{2\left(1-(-1)^k\right)}{k\pi}\sin(kx)}$$

convergence discussion



define $f_n(x) = \sum_{k=0}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$

nth partial sum

For fixed x not a multiple of π,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

since $f(x)$ is not continuous at multiples of π

The Fourier series converges <u>pointwise</u>,

but not <u>uniformly</u>.

3. Find the Fourier series of $f(x) = 3x - 2$. Then plot $f(x)$ together with partial sums of its Fourier series.

We didn't do this in class, but here is the solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (3-2x)\,dx = 6$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (3-2x)\cos(kx)\,dx = \frac{6\sin(k\pi)}{k\pi} = 0 \quad \text{for } k = 1, 2, 3, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (3-2x)\sin(kx)\,dx = \frac{4(-1)^k}{k} \quad \text{for } k = 1, 2, 3, \ldots$$

Thus, the Fourier series is: $3 - 2x \sim 3 + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k}\sin(kx)$

4. What property of $f(x)$ guarantees that its Fourier series consists only of sine terms?

$f(x)$ is an even function

That is: $f(-x) = f(x)$

5. What property of $f(x)$ guarantees that its Fourier series consists of only cosine terms?

$f(x)$ is an odd function

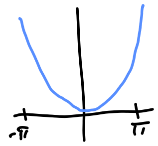
That is: $f(-x) = -f(x)$

6. Based on the Fourier series that you have seen so far, make a conjecture about the points at which the Fourier series for $f(x)$ converges to $f(x)$.

Fourier Series

Math 330

1. Match the following 2π−periodic functions with their Fourier series *without* solving for the coefficients.



(a) $f(x) = x^2$ for $x \in [-\pi, \pi]$

even

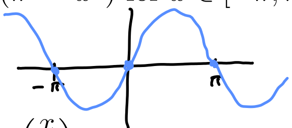
avg. is pos

$f(-x) = -f(x)$

(b) $f(x) = x(\pi^2 - x^2)$ for $x \in [-\pi, \pi]$

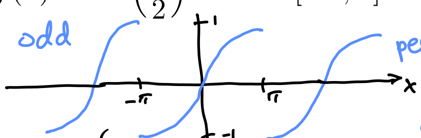
odd

periodic ext is continuous

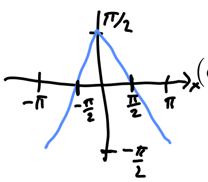


(c) $f(x) = \sin\left(\frac{x}{2}\right)$ for $x \in [-\pi, \pi]$

odd



periodic extension is not continuous



(d) $f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \le x < 0 \\ \frac{\pi}{2} - x, & 0 \le x \le \pi \end{cases}$

even

avg = zero

I. $f(x) = \dfrac{8}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{k(-1)^k}{1 - 4k^2} \sin(kx)$ $\approx \frac{1}{4k}$ odd converges slowly

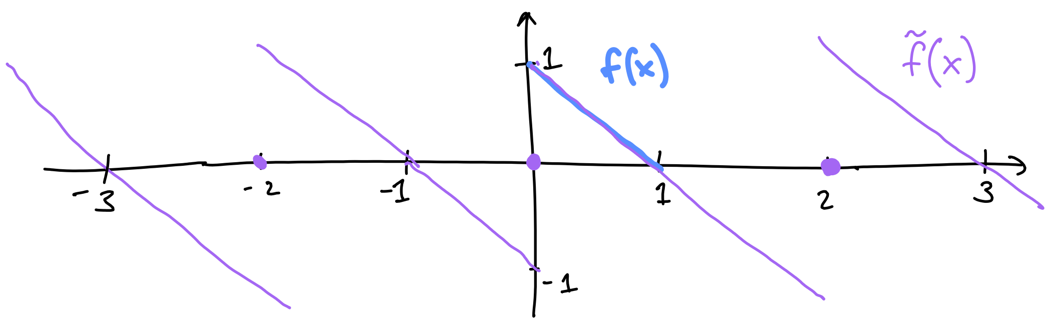
II. $f(x) = \dfrac{\pi^2}{3} + 4 \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)$ even

III. $f(x) = -12 \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx)$ $\approx \frac{1}{k^3}$ odd converges quickly

IV. $f(x) = \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k^2} \cos(kx)$ even

2. Let $f(x) = 1 - x$ be defined on $x \in [0, 1]$.

(a) Sketch $\tilde{f}(x)$, the odd 2-periodic extension of $f(x)$. That is, $\tilde{f}(x)$ should be an odd function with period 2.



(b) If you were to write a trigonometric series that converges $\tilde{f}(x)$, what terms would you put in this series?

~~sin(x) has period 2π~~

sin(πx) has period 2 also: cos(πx)

sin(2πx) has period 1 cos(2πx)

sin(3πx) has period 2/3 cos(3πx)

use these trig functions!

(c) Compute the coefficients for your trigonometric series from part (b). Then plot some partial sums of this series together with $\tilde{f}(x)$.

We will do this next time...

(d) Sketch $\hat{f}(x)$, the even 2-periodic extension of $f(x)$. That is, $\hat{f}(x)$ should be an even function with period 2.

(e) If you were to write a trigonometric series that converges $\hat{f}(x)$, what terms would you put in this series?

(f) Compute the coefficients for your trigonometric series from part (e). Then plot some partial sums of this series together with $\hat{f}(x)$.