Wave Equation: d'Alembert's Solution

Math 330

1. Show that if F and G are any C^2 functions, then $u(x,t) = F(x+ct) + G(x-ct)$ solves the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

2. Consider the wave equation on an *infinite string*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, t \geq 0$$

with initial conditions $u(0,x) = f(x)$ and $\frac{\partial u}{\partial t}(0,x) = g(x)$.

(a) Consider the spacetime variables $\xi = x + ct$ and $\eta = x - ct$. Show that the PDE $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ transforms into $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ with these new variables.

From last class:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2}$$

Then $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ becomes $c^2 \cancel{\frac{\partial^2 u}{\partial \xi^2}} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \cancel{\frac{\partial^2 u}{\partial \eta^2}} = c^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$

so $0 = 4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta}$, which implies $\boxed{0 = \frac{\partial^2 u}{\partial \xi \partial \eta}}$.

(b) Integrate twice to show that $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ is solved by $u(\xi, \eta) = p(\xi) + q(\eta)$.

Integrate w.r.t. η: $\displaystyle\int \frac{\partial^2 u}{\partial \eta \partial \xi}\, d\eta = \int 0 \, d\eta$

$$\frac{\partial u}{\partial \xi} = r(\xi) \quad \leftarrow \text{some function of } \xi$$

Integrate w.r.t. ξ: $\displaystyle\int \frac{\partial u}{\partial \xi}\, d\xi = \int r(\xi)\, d\xi$

$$u(\xi, \eta) = \underset{\substack{\uparrow \\ \text{antiderivative of } r(\xi)}}{p(\xi)} + q(\eta) \quad \leftarrow \text{some function of } \eta$$

(c) Transform your solution $p(\xi) + q(\eta)$ back to the original coordinates x and t. Can you give a physical interpretation of this solution?

Since $\xi = x + ct$ and $\eta = x - ct$,

the solution becomes $u(t,x) = \underbrace{p(x+ct)}_{\text{wave traveling left}} + \underbrace{q(x-ct)}_{\text{wave traveling right}}$.

$$\frac{\partial u}{\partial t} = c \cdot p'(x+ct) - c \cdot q'(x-ct)$$

$$\frac{\partial u}{\partial t}(0,x) = c \cdot p'(x) - c \cdot q'(x)$$

(d) Substitute your solution into the two initial conditions. Integrate the second expression from 0 to x. Use algebra to solve for functions p and q.

Initial conditions: $u(0,x) = f(x) = p(x) + q(x)$ and $\frac{\partial u}{\partial t}(0,x) = g(x) = \underbrace{c p'(x) - c q'(x)}_{\text{integrate}}$

$$\boxed{G(x) = c \cdot p(x) - c\, q(x)}$$

where $G'(x) = g(x)$

so G is an antideriv. of g

$$G(x) = \int g(x)\,dx$$

We have
$$\boxed{\begin{aligned} p(x) + q(x) &= f(x) \\ p(x) - q(x) &= \tfrac{1}{c} G(x) \end{aligned}}$$

Add to obtain: $2p(x) = f(x) + \frac{1}{c} G(x)$

Subtract to obtain: $2q(x) = f(x) - \frac{1}{c} G(x)$

So: $p(x) = \frac{1}{2} f(x) + \frac{1}{2c} G(x)$ and $q(x) = \frac{1}{2} f(x) - \frac{1}{2c} G(x)$

(e) Manipulate your expressions to arrive at the solution

$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau.$$

This is known as *D'Alembert's solution.*

$$u(t,x) = \underbrace{p(x+ct)}_{} + \underbrace{q(x-ct)}_{}$$

$$= \underbrace{\tfrac{1}{2} f(x+ct) + \tfrac{1}{2c} G(x+ct)}_{} + \underbrace{\tfrac{1}{2} f(x-ct) - \tfrac{1}{2c} G(x-ct)}_{}$$

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\underbrace{\left[G(x+ct) - G(x-ct)\right]}_{}$$

FTC

$$\int_a^b g(\tau)\,d\tau = G(b) - G(a)$$

$$\boxed{u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau}$$

3. Consider the following wave equation on an infinite medium ($-\infty < x < \infty$):

init. pos. $f(x)$ init. vel. zero

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \boxed{u(0,x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}}, \qquad \underbrace{\frac{\partial u}{\partial t}(0,x) = 0}_{g(x)}$$

(a) Sketch what you think the solution should look like at $t = 0, t = 1$, and $t = 2$ (without actually finding the solution yet).

(b) Find d'Alembert's solution to this wave equation.

$c = 1$

$$u(t,x) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \overset{0}{\cancel{g(\tau)\,d\tau}}$$

$$u(t,x) = \frac{1}{2}\begin{cases} x-t, & 0 < x-t < 1 \\ 2-(x-t), & 1 \leq x-t < 2 \\ 0, & \text{otherwise} \end{cases} + \frac{1}{2}\begin{cases} x+t, & 0 < x+t < 1 \\ 2-(x+t), & 1 \leq x+t < 2 \\ 0, & \text{otherwise} \end{cases}$$

(c) Use software to plot d'Alembert's solution at $t = 0$, $t = 1$, and $t = 2$ to verify your earlier plot.

4. Find d'Alembert's solution to the following wave equation on an infinite medium:

$c = \frac{1}{2}$

$$\frac{\partial^2 u}{\partial t^2} = \overset{c^2 = \frac{1}{4}}{\frac{1}{4}}\frac{\partial^2 u}{\partial x^2}, \qquad \underbrace{u(0,x) = 0}_{f(x)=0}, \qquad \frac{\partial u}{\partial t}(0,x) = \underbrace{\cos(x)}_{g(x) = \cos(x)}$$

Use software to plot the wave profile over time.

$$u(t,x) = \int_{x-\frac{1}{2}t}^{x+\frac{1}{2}t} \cos(x)\,dx$$

$$u(t,x) = \sin\left(x + \frac{1}{2}t\right) - \sin\left(x - \frac{1}{2}t\right)$$

5. For each of the following wave equations on an infinite medium, sketch (by hand) the wave profiles at $t = 0$, $t = 1$, and $t = 2$ without solving the equations. Check your sketch using software.

(a) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(0,x) = 0$$

(b) $$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = \begin{cases} x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(0,x) = 0$$