Wave Equation: d'Alembert's Solution

Math 330

1. Show that if F and G are any C^2 functions, then $u(x,t) = F(x+ct) + G(x-ct)$ each solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial dx^2}$.

2. Consider the wave equation on an *infinite string*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial dx^2}, \qquad -\infty < x < \infty, t \geq 0$$

with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$.

(a) Consider the spacetime variables $\xi = x + ct$ and $\eta = x - ct$. Show that the PDE $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ transforms into $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ with these new variables.

(b) Integrate twice to show that $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ is solved by $u(\xi, \eta) = p(\xi) + q(\eta)$.

(c) Transform your solution $p(\xi) + q(\eta)$ back to the original coordinates x and t. Can you give a physical interpretation of this solution?

(d) Substitute your solution into the two initial conditions. Integrate the second expression from 0 to x. Use algebra to solve for functions p and q.

(e) Manipulate your expressions to arrive at the solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\ d\tau.$$

This is known as *D'Alembert's solution.*

3. Consider the following wave equation on an infinite medium ($-\infty < x < \infty$):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

(a) Sketch what you think the solution should look like at $t = 0, t = 1$, and $t = 2$ (without actually finding the solution yet).

(b) Find d'Alembert's solution to this wave equation.

(c) Use software to plot d'Alembert's solution at $t = 0$, $t = 1$, and $t = 2$ to verify your earlier plot.

4. Find d'Alembert's solution to the following wave equation on an infinite medium:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \cos(x)$$

Use software to plot the wave profile over time.

5. For each of the following wave equations on an infinite medium, sketch (by hand) the wave profiles at $t = 0$, $t = 1$, and $t = 2$ without solving the equations. Check your sketch using software.

(a) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

(b) $$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$