Waves and Transport

Math 330

1. Consider the differential equation:

$$\frac{\partial u}{\partial t} = 0$$

(a) If $u = u(t)$ is a function of t alone, then what are all solutions to the differential equation?

(b) If $u = u(t, x)$, then what are all solutions to the differential equation?

(c) Are all solutions to this differential equation constant in t? That is, if $u(t, x)$ is it a solution, does it follow that $u(t_1, x) = u(t_2, x)$ if $t_1 \neq t_2$?

2. Consider the *transport equation*

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is some constant.

(a) Introduce the *characteristic variable* $\xi = x - ct$. That is, $u(t,x) = v(t, x-ct) = v(t,\xi)$. Use the multivariable chain rule to explain why

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi} \qquad \text{and} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}.$$

(b) How does the transport equation simplify when expressed in terms of the characteristic variable? What functions v satisfy this equation?

(c) Transform your solution v back to the original variables t and x. Can you give a physical interpretation of this solution? (*Hint*: What is the role of t, x, and c?)

3. Find the solution to the initial value problem

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0, \qquad u(0,x) = \frac{1}{1+x^2}.$$

4. Now consider the differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$$

where a is a positive constant and c is any constant.

(a) Introduce the change of variable $\xi = x - ct$ as before. How does this simplify the differential equation?

(b) Multiply your equation by the integrating factor e^{at}. Show that $\frac{\partial}{\partial t}\left(e^{at}v\right) = 0$. What does this imply about $e^{at}v$?

(c) Let $f(\xi)$ be a C^1 function and suppose $e^{at}v = f(\xi)$. Solve for v and transform your solution back to the original variables t and x.

(d) What initial value problem have you now solved? Give a physical interpretation for your solution.

5. Find the solution to the initial value problem

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} + u = 0, \qquad u(0,x) = \frac{1}{1+x^2}.$$