FINITE DIFFERENCE APPROXIMATIONS

ERROR

For f':

forward: $f'(x) \approx \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ $O(\Delta x)$

backward: $f'(x) \approx \dfrac{f(x-\Delta x) - f(x)}{-\Delta x}$ $O(\Delta x)$

note this negative sign →

centered: $f'(x) \approx \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$ $O(\Delta x^2)$

For f'':

centered: $f''(x) \approx \dfrac{f(x-\Delta x) - 2f(x) + f(x+\Delta x)}{(\Delta x)^2}$ $O(\Delta x^2)$

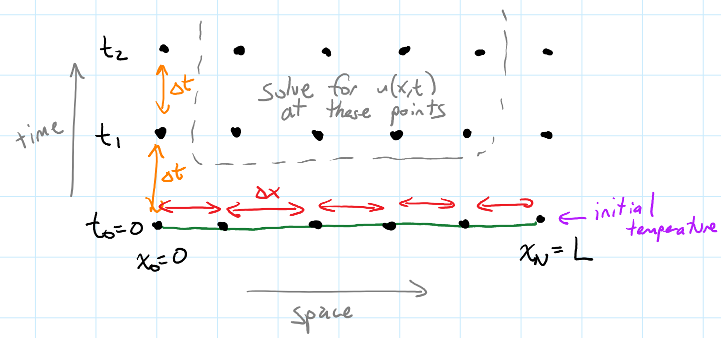
other approximations?

HEAT EQUATION: $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$

initial condition $u(x,0) = f(x)$

boundary conditions: $u(0,t) = u(L,t) = 0$

We will approximate $u(x,t)$ on a rectangular grid of points.



NOTATION: Let $u_j^{(m)}$ be the approximate solution $u(x_j, t_m)$

i.e., the approx. temperature at position x_j, time t_m

Forward diff. in time: $\dfrac{\partial u}{\partial t}(x,t) \approx \dfrac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$

Centered diff in space: $\dfrac{\partial^2 u}{\partial x^2}(x,t) \approx \dfrac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$

Heat equation becomes: $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$

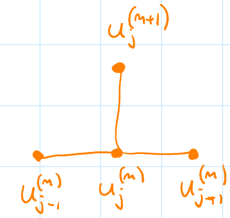
$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \approx k \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

For $x = x_j$ and $t = t_m$: $$\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = k \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{\Delta x^2}$$

Solve for $u_j^{(m+1)}$:

$$u_j^{(m+1)} = u_j^{(m)} + \frac{k\Delta t}{\Delta x^2}\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right)$$

let $s = \frac{k\,\Delta t}{\Delta x^2}$



Stencil diagram

Finite difference equation for the EXPLICIT SCHEME:

$$\boxed{u_j^{(m+1)} = s\,u_{j+1}^{(m)} + (1-2s)\,u_j^{(m)} + s\,u_{j-1}^{(m)}}$$

COMPUTATION:

let $u^{(m)}$ be the vector of approximations at interior points at time t_m

$$u^{(m)} = \begin{bmatrix} u_1^{(m)} \\ u_2^{(m)} \\ \vdots \\ u_{N-1}^{(m)} \end{bmatrix}$$

Let A be the $(N-1)\times(N-1)$ matrix:

$$A = \begin{bmatrix} 1-2s & s & 0 & 0 & 0 & 0 & \cdots & 0 \\ s & 1-2s & s & 0 & 0 & 0 & \cdots & 0 \\ 0 & s & 1-2s & s & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & s & 1-2s \end{bmatrix}$$

Then: $u^{(m+1)} = A\,u^{(m)}$

$$\begin{bmatrix} u_1^{(m+1)} \\ u_2^{(m+1)} \\ \vdots \\ u_{N-1}^{(m+1)} \end{bmatrix} = \begin{bmatrix} 1-2s & s & & & 0 \\ s & 1-2s & s & & \\ & s & 1-2s & s & \\ 0 & & & \ddots & \end{bmatrix} \begin{bmatrix} u_1^{(m)} \\ u_2^{(m)} \\ \vdots \\ u_{N-1}^{(m)} \end{bmatrix}$$

Let's compute! Work through the Mathematica notebook to see the explicit scheme in action.

Stability Analysis

Math 330

We have observed that the "explicit scheme" for finding approxiate solutions to the heat equation sometimes produces solutions that exhibit crazy behavior over time. To understand this, we will examine the effect of the explicit scheme on simple functions.

Recall that the explicit scheme uses the partial difference equation:

$$u_j^{(m+1)} = s u_{j+1}^{(m)} + (1-2s)u_j^{(m)} + s u_{j-1}^{(m)}$$

1. Suppose that, at some time t_m, the approximate solution is given by $u(x, t_m) = e^{i\alpha x}$, for some $\alpha \in \mathbb{R}$. Substitute $u_j^{(m)} = e^{i\alpha x_j}$ into the partial difference equation and write the result:

2. Remembering that $x_{j-1} = x_j - \Delta x$ and $x_{j+1} = x_j + \Delta x$, manipulate your equation from #1 to obtain

$$u_j^{(m+1)} = u_j^{(m)} \left[1 - 2s + s\left(e^{i\alpha\Delta x} + e^{-i\alpha\Delta x}\right)\right].$$

3. Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to rewrite your equation as

$$u_j^{(m+1)} = u_j^{(m)} \left[1 - 2s\left(1 - \cos(\alpha\Delta x)\right)\right].$$

4. Let $Q = 1 - 2s\,(1 - \cos(\alpha\Delta x))$. The partial difference equation then becomes

$$u_j^{(m+1)} = Qu_j^{(m)}.$$

What happens to $u_j^{(m)}$ if $|Q| < 1$? What happens if $|Q| > 1$?

5. Explain why $Q \leq 1$.

6. Show that $-1 < Q$ is equivalent to that $\dfrac{1}{1 - \cos(\alpha\Delta x)} > s$. You may assume $\cos(\alpha\Delta x) \neq 1$.

7. Explain why $-1 < Q$ if $s < \frac{1}{2}$.

8. Why might the solution be badly behaved if $s > \frac{1}{2}$?

9. If we require that $s < \frac{1}{2}$, what restrictions does this impose on Δx and Δt?

Investigate the effect of the explicit scheme on simple functions.

Suppose: $u(x, t_j) = e^{i\alpha x}$ for some $\alpha \in \mathbb{R}$

so $u_j^{(m)} = e^{i\alpha x_j}$ and:

$$u_j^{(m+1)} = s\, e^{i\alpha x_{j+1}} + (1-2s)\, e^{i\alpha x_j} + s e^{i\alpha x_{j-1}}$$

Note:
$x_{j-1} = x_j - \Delta x$
$x_{j+1} = x_j + \Delta x$

$$= s\, e^{i\alpha(x_j+\Delta x)} + (1-2s)\, e^{i\alpha x_j} + s e^{i\alpha(x_j - \Delta x)}$$

$$= e^{i\alpha x_j}\left[s e^{i\alpha\Delta x} + 1 - 2s + s e^{-i\alpha\Delta x}\right]$$

$$= u_j^{(m)}\left[1 - 2s + s\left(e^{i\alpha\Delta x} + e^{-i\alpha\Delta x}\right)\right]$$

$$e^{i\alpha\Delta x} + e^{-i\alpha\Delta x} = \left(\cos(\alpha\Delta x) + \cancel{i\sin(\alpha\Delta x)}\right) + \left(\cos(\alpha\Delta x) - \cancel{i\sin(\alpha\Delta x)}\right) = 2\cos(\alpha\Delta x)$$

$$= u_j^{(m)}\left[1 - 2s + s\left(2\cos(\alpha\Delta x)\right)\right]$$

$$u_j^{(m+1)} = u_j^{(m)}\underbrace{\left[1 - 2s\left(1 - \cos(\alpha\Delta x)\right)\right]}_{\text{let this be } Q \text{ — the "magnification factor"}}$$

Then $u_j^{(m+1)} = Q\, u_j^{(m)}$, and $u_j^{(m+p)} = Q^p u_j^{(m)}$.

If $|Q| > 1$, then the approximation grows exponentially.

If $|Q| < 1$, then the approximation converges.

We want: $-1 < Q \leq 1$

$$-1 < 1 - 2s\left(1 - \cos(\alpha\Delta x)\right) \leq 1$$

holds automatically, since $s > 0$ and $1 \geq \cos(\alpha\Delta x)$

$$-1 < 1 - 2s\left(1 - \cos(\alpha\Delta x)\right)$$

$$-2 < -2s\left(1 - \cos(\alpha\Delta x)\right)$$

$$1 > s\left(1 - \cos(\alpha\Delta x)\right)$$

Want: $$\frac{1}{1 - \cos(\alpha\Delta x)} > s$$

Since $\dfrac{1}{1 - \cos(\alpha\Delta x)} \geq \dfrac{1}{2}$, we will have $-1 < Q$ if $s < \frac{1}{2}$.

The explicit scheme is **STABLE** if $s < \frac{1}{2}$, but **UNSTABLE** if $s > \frac{1}{2}$.

Since $s = \dfrac{k\Delta t}{(\Delta x)^2}$, we want $\dfrac{k\Delta t}{(\Delta x)^2} < \dfrac{1}{2}$, so $\Delta t < \dfrac{(\Delta x)^2}{2k}$.

→ Time step must be <u>much</u> smaller than spatial step.