STURM-LIOUVILLE APPLICATIONS

1. $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2}$, $u(0,t)=0$, $\underbrace{\frac{\partial u}{\partial x}(L,t) = -h\,u(L,t)}_{\text{boundary condition of the "third kind"}}$

(a) Heat flow is proportional to temperature difference

$$\underset{\text{heat flow}}{\frac{\partial u}{\partial x}(L,t)} = -h\,\underbrace{u(L,t) - 0}_{\text{temperature difference, if the ambient temperature is zero}}$$

physical context: we expect $h>0$

(b) $u(x,t) = G(t)\,\phi(x)$ separates: $G' = -\lambda k G$ and

$$\boxed{\begin{aligned} &\phi'' = -\lambda\phi \\ &\phi(0)=0 \\ &\phi'(L)+h\,\phi(L)=0 \end{aligned}}$$

Sturm-Liouville Problem

$p=1$, $q=0$, $\sigma=1$

(c) Rayleigh Quotient:

$$\lambda = \frac{-p\phi\phi'\big|_a^b + \int_a^b \left(p(\phi')^2 - q\phi^2\right)dx}{\int_a^b \phi^2\sigma\,dx} = \frac{-\phi\phi'\big|_0^L + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}$$

$$= \frac{h\,\phi^2(L) + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}$$

So $\lambda>0$ if $h>0$.

If $h<0$, then there might be negative eigenvalues.

(d) If $\lambda>0$, then the solution is $\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$.

Boundary conditions: $\phi(0)=0$ implies $c_1=0$, so $\phi(x) = c_2\sin(\sqrt{\lambda}x)$,

$\phi'(x) = \sqrt{\lambda}\,c_2\cos(\sqrt{\lambda}x)$

$\phi'(L) = -h\,\phi(L)$ implies $\sqrt{\lambda}\,c_2\cos(L\sqrt{\lambda}) = -h\,c_2\sin(\sqrt{\lambda}\,x)$

or $c_2\left(\sqrt{\lambda}\cos(L\sqrt{\lambda}) + h\sin(L\sqrt{\lambda})\right) = 0$

We don't want $c_2=0$, so we need $\sqrt{\lambda}\cos(L\sqrt{\lambda}) + h\sin(L\sqrt{\lambda}) = 0$

or $\boxed{\tan(L\sqrt{\lambda}) = \frac{-\sqrt{\lambda}}{h}}$

(e) Let $t=\sqrt{\lambda}$, and plot $y=\frac{-t}{h}$ and $y=\tan(tL)$.

Intersection points of the graphs give eigenvalues.

(f) Large eigenvalues are close to the vertical asymptotes of $\tan(tL)$.

As $n\to\infty$, $\sqrt{\lambda_n}\,L \longrightarrow (n-\frac{1}{2})\pi$.

(g) See section 5.8 in the text, especially pages 196-199.

2. $$r^2\frac{d^2f}{dr^2}+r\frac{df}{dr}+(\lambda r^2-m^2)f=0 \qquad (*)$$

(a) Divide by r: $$r\frac{d^2f}{dr^2}+\frac{df}{dr}+\left(\lambda r-\frac{m^2}{r}\right)f=0$$

$$\frac{d}{dr}\left(r\frac{df}{dr}\right)-\frac{m^2}{r}f+\lambda r f=0$$

This is a S-L equation with $x=r$, $p(r)=r$, $q(r)=\frac{-m^2}{r}$, $\sigma(r)=r$.

Note: $r=0$ is a singular point: solutions might be badly behaved at $r=0$ (maybe vertical asymptote).

(b) Change of variables: $z=\sqrt{\lambda}\,r$ or $r=\frac{z}{\sqrt{\lambda}}$

So $r=\frac{z}{\sqrt{\lambda}}$ and $\frac{df}{dr}=\frac{df}{dz}\cdot\frac{dz}{dr}=\frac{df}{dz}\sqrt{\lambda}$ or $\frac{df}{dz}=\frac{df}{dr}\frac{1}{\sqrt{\lambda}}$

Similarly, $\frac{d^2f}{dr^2}=\frac{d^2f}{dz^2}\lambda$

So equation $(*)$ becomes:

$$\left(\frac{z}{\sqrt{\lambda}}\right)^2\frac{d^2f}{dz^2}\lambda+\frac{z}{\sqrt{\lambda}}\frac{df}{dz}\sqrt{\lambda}+\left(\lambda\left(\frac{z}{\sqrt{\lambda}}\right)^2-m^2\right)f=0$$

$$\boxed{z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+(z^2-m^2)f=0}$$

BESSEL'S EQUATION OF ORDER m

(c) The Bessel functions of order m solve Bessel's equation of order m:

$J_m(z)$ is the Bessel function of the first kind of order m, and is finite at $z=0$. Mathematica: BesselJ[m, z]

$Y_m(z)$ is the Bessel function of the second kind of order m, and has a vertical asymptote at $z=0$. Mathematica: BesselY[m, z]

(d) Boundary conditions: $|f(0)| < \infty$ ← solution is bounded at the origin

$\underset{r=a}{f(a)} = 0$ ← boundary of the drum is fixed

General solution: $f(z) = c_1 J_m(z) + c_2 Y_m(z)$

blows up at $z=0$

$|f(0)| < \infty$ implies that $c_2 = 0$

So $f(z) = c_1 J_m(z)$

Condition $\underset{r=a}{f(a)} = 0$ implies $0 = c_1 \underbrace{J_m(\sqrt{\lambda}\, a)}_{\text{Want: } J_m(\sqrt{\lambda}\, a) = 0}$

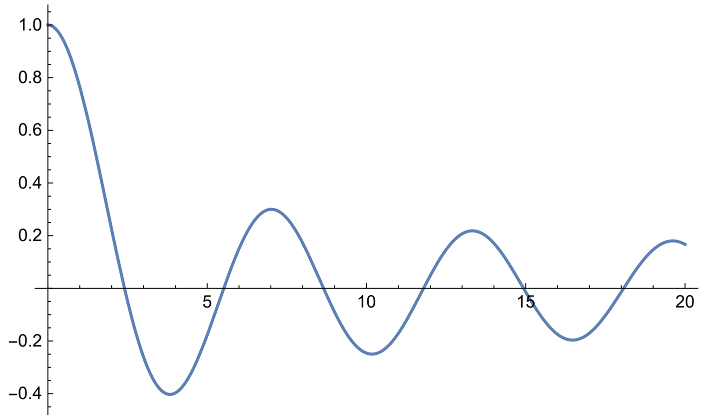
So choose λ so that $\sqrt{\lambda}\, a$ is a zero of J_m

to be continued...

Bessel function of the first kind of order 0:

In[19]:= **Plot[BesselJ[0, z], {z, 0, 20}]**

Out[19]=



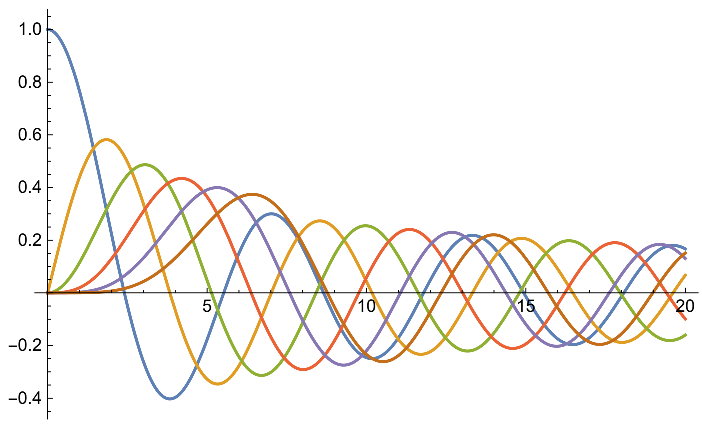
Bessel functions of the first kind of order 0, 1, ..., 5:

In[21]:= **bjs = Table[BesselJ[m, z], {m, 0, 5}]**

Out[21]= {BesselJ[0, z], BesselJ[1, z], BesselJ[2, z], BesselJ[3, z], BesselJ[4, z], BesselJ[5, z]}

In[22]:= **Plot[bjs, {z, 0, 20}]**

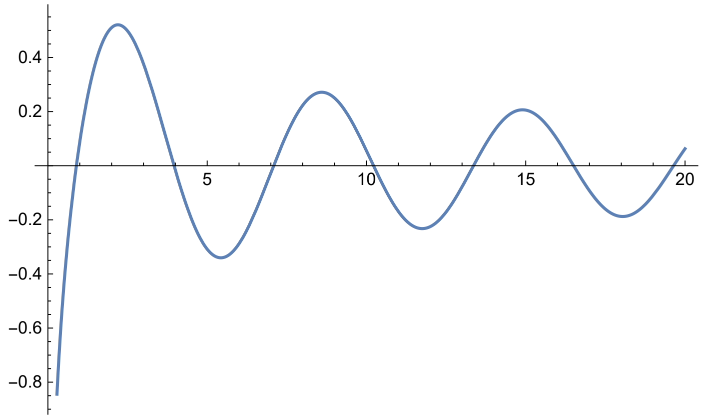
Out[22]=



Bessel function of the second kind of order 0:

In[23]:=
```
Plot[BesselY[0, z], {z, 0, 20}]
```

Out[23]=



Bessel functions of the second kind of order 0, 1, 2, ..., 5

In[24]:=
```
bys = Table[BesselY[m, z], {m, 0, 5}]
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Out[24]=
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{BesselY[0, z], BesselY[1, z], BesselY[2, z], BesselY[3, z], BesselY[4, z], BesselY[5, z]}
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In[25]:=
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Plot[bys, {z, 0, 20}]
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Out[25]=