STURM-LIOUVILLE OPERATOR: $L(\phi) = \frac{d}{dx}\left[p \frac{d\phi}{dx}\right] + q\phi$

For any functions u and v:

LAGRANGE'S IDENTITY: $uL(v) - vL(u) = \frac{d}{dx}\left[u \cdot p \frac{dv}{dx}\right] - \frac{d}{dx}\left[v \cdot p \frac{du}{dx}\right]$

GREEN'S IDENTITY: $\int_a^b \left(uL(v) - vL(u)\right) dx = \left[p\left(u \cdot \frac{dv}{dx} - v \cdot \frac{du}{dx}\right)\right]_a^b$

If $\int_a^b \left(uL(v) - vL(u)\right) dx = 0$ for all u, v (in some set of functions), then L is a **SELF-ADJOINT OPERATOR.**

THEOREM: If u and v both satisfy the same regular S-L boundary conditions, then $\int_a^b \left(uL(v) - vL(u)\right) dx = 0$.

proof: Let

$$\left.\begin{aligned} \beta_1 u(a) + \beta_2 u'(a) = 0 \quad &\text{and} \quad \beta_1 v(a) + \beta_2 v'(a) = 0 \\ \beta_3 u(b) + \beta_4 u'(b) = 0 \quad & \quad\quad\;\; \beta_3 v(b) + \beta_4 v'(b) = 0 \end{aligned}\right] \text{regular S-L boundary conditions}$$

Green's identity:

$$\int_a^b \left(uL(v) - vL(u)\right) dx = \left[p\left(u \cdot v' - v \cdot u'\right)\right]_a^b$$

$$= p(b)\left(u(b)\underbrace{v'(b)}_{\downarrow} - v(b)\underbrace{u'(b)}_{\downarrow}\right) - p(a)\left(u(a)\underbrace{v'(a)}_{\downarrow} - v(a)u'(a)\right)$$

$$= p(b)\left(u(b)\left(\frac{-\beta_3 v(b)}{\beta_4}\right) - v(b)\left(\frac{-\beta_3 u(b)}{\beta_4}\right)\right) - p(a)\left(u(a)\left(\frac{-\beta_1 v(a)}{\beta_2}\right) - v(a)\left(\frac{-\beta_1 u(a)}{\beta_2}\right)\right)$$

$$= 0$$

RAYLEIGH QUOTIENT

$$\lambda = \frac{-p\phi \left.\frac{d\phi}{dx}\right|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right] dx}{\int_a^b \phi^2 \sigma \, dx}$$

MINIMIZATION PRINCIPLE: The minimum value of the Rayleigh Quotient over all continuous functions satisfying the boundary conditions is the smallest eigenvalue.

1. $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\phi(0)=0$, $\frac{d\phi}{dx}(1)=0$

(a) Try a quadratic: $\phi_T(x) = x^2 - 2x$

(b) $$\lambda = \frac{0 + \int_0^1 (2x-2)^2\,dx}{\int_0^1 (x^2-2x)^2\,dx} = \boxed{\frac{5}{2}}$$ ← The smallest eigenvalue must be not greater than $\frac{5}{2}$.

Another function? Try: $\phi_T(x) = \sin\left(\frac{\pi}{2}x\right)$

Then: $$\lambda = \frac{\int_0^1 \left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\right)^2 dx}{\int_0^1 \sin^2\left(\frac{\pi}{2}x\right)dx} = \frac{\pi^2}{4} = \boxed{2.467}$$ ← The smallest eigenvalue must be not greater than this.

Solution: $\phi = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

$\phi(0)=0$ implies $A=0$, $\phi'(1)=0$ implies $\sqrt{\lambda} = n\pi + \frac{\pi}{2}$ for $n=0,1,2,\ldots$

2. — skip

3. $\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}$, $u(0,t)=u(L,t)=0$, $u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=g(x)$

(a) The equation describes vibration of a string with nonuniform density, fixed endpoints, and constant tension.

(b) $u(x,t) = \phi(x)h(t)$ so $\rho\phi h'' = T_0\phi'' h$, and thus $\frac{h''}{h} = \frac{T_0}{\rho}\frac{\phi''}{\phi} = -\lambda$

This yields $h'' = -\lambda h$ and $T_0\phi'' + \lambda\rho\phi = 0$ with $\phi(0)=\phi(L)=0$

(c) Rayleigh quotient: $$\lambda = \frac{0 + \int_0^L \left[T_0(\phi')^2\right]dx}{\int_0^L \phi^2\rho\,dx} > 0$$ since all quantities are nonnegative and ϕ cannot be the zero function

(d) $h_n(t) = a_n\cos(t\sqrt{\lambda_n}) + b_n\sin(t\sqrt{\lambda_n})$

So $$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos(t\sqrt{\lambda_n}) + b_n\sin(t\sqrt{\lambda_n})\right)\phi_n(x)$$

(e) smallest eigenvalue: $\lambda_1 = \min_u \dfrac{T_0 \int_0^L (u')^2 dx}{\int_0^L u^2 \rho \, dx}$

bounds on density: $\rho_{min} \le \rho(x) \le \rho_{max}$

$$\text{So: } \rho_{min} \int_0^L u^2 dx \le \int_0^L \rho(x) u^2 dx \le \rho_{max} \int_0^L u^2 dx$$

Thus:

$$\frac{T_0}{\rho_{max}} \boxed{\min_u \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 dx}} \le \lambda_1 \le \frac{T_0}{\rho_{min}} \boxed{\min_u \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 dx}}$$

These represent the smallest eigenvalue of $\phi'' + \lambda \phi = 0$ with $\phi(0) = \phi(L) = 0$, and this smallest eigenvalue is $\left(\frac{\pi}{L}\right)^2$.

Therefore:

$$\frac{T_0}{\rho_{max}} \left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{T_0}{\rho_{min}} \left(\frac{\pi}{L}\right)^2$$

INTERPRETATION: The lowest frequency of vibration of a nonuniform string lies between the lowest frequencies of strings with constant densities that bound the nonuniform density above and below.

(See section 5.7 in the text for more details.)