EIGENFUNCTION EXPANSION

ASSOCIATED HOMOGENEOUS PDE:

1. (a) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $\quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

Let $u(x,t) = \phi(x)G(t)$, then separation of variables produces

$$\frac{dG}{dt} = -\lambda k G \quad \text{and} \quad \underbrace{\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{with} \quad \phi'(0) = \phi'(L) = 0}_{\text{BVP}}$$

eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$, eigenfunctions $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$

(b) NONHOMOGENEOUS: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + 1$

Suppose $u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\left(\frac{n\pi}{L}x\right)$ is a solution

unknown functions

$$\frac{\partial u}{\partial t} = A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\left(\frac{n\pi}{L}x\right)$$ ← OK, since we assume $\frac{\partial u}{\partial t}$ is piecewise smooth

$$\frac{\partial u}{\partial x} = -\sum_{n=1}^{\infty}\frac{n\pi}{L}A_n(t)\sin\left(\frac{n\pi}{L}x\right)$$ ← OK, since the Fourier cosine series of a continuous function can be differentiated

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 A_n(t)\cos\left(\frac{n\pi}{L}x\right)$$ ← OK, since $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ and we assume all derivatives of u are continuous

(c)
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + 1$$

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\left(\frac{n\pi}{L}x\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 A_n(t)\cos\left(\frac{n\pi}{L}x\right) + 1$$

$A_0'(t) = 1$ and $A_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 A_n(t)$

Initial conditions: $u(x,0) = f(x)$

so $f(x) = u(x,0) = A_0(0) + \sum_{n=1}^{\infty} A_n(0)\cos\left(\frac{n\pi}{L}x\right)$

Then: $A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$

(d) Solve for $A_0(t)$: $A_0(t) = t + C_0$

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so $A_0(t) = t + \frac{1}{L}\int_0^L f(x)\,dx$

Solve for $A_n(t)$: $A_n(t) = C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

so $A_n(t) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

Finally, the solution to the PDE:

$$\boxed{u(x,t) = t + \frac{1}{L}\int_0^L f(x)\,dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)}$$

2. $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t}x$$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ — boundary conditions

$u(x,0) = 1$ — initial condition

Associated homogeneous eq. is same as before, with eigenfunctions $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ and eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

Suppose $\boxed{u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\left(\frac{n\pi}{L}x\right)}$ solves the nonhomogeneous PDE.

Plug in: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t}x$ — write x as a Fourier cosine series

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\left(\frac{n\pi}{L}x\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 A_n(t)\cos\left(\frac{n\pi}{L}x\right) + e^{-t}\left(\frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L\left((-1)^n - 1\right)}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\right)$$

$n=0$: $A_0'(t) = e^{-t}\frac{L}{2} \implies A_0(t) = -\frac{L}{2}e^{-t} + C_0$

$A_0(0) = 1$, so $\boxed{A_0(t) = -\frac{L}{2}e^{-t} + \left(1 + \frac{L}{2}\right)}$

INITIAL CONDITIONS

$u(x,0) = 1$

$1 = A_0(0) + \sum_{n=1}^{\infty} A_n(0)\cos\left(\frac{n\pi}{L}x\right)$

$\Downarrow$

$A_0(0) = 1, \quad A_n(0) = 0$

$n>0$: $A_n'(t) = \underbrace{-k\left(\frac{n\pi}{L}\right)^2}_{J} A_n(t) + \underbrace{\frac{2L\left((-1)^n-1\right)}{n^2\pi^2}}_{Q} e^{-t}$ and $A_n(0)=0$

$e^{-Jt}A_n'(t) - J A_n(t) e^{-Jt} = Q e^{-t} e^{-Jt}$ integrating factor: $\mu(t) = e^{\int -J\,dt} = e^{-Jt}$

$$\int \frac{d}{dt}\left(A_n(t)\, e^{-Jt}\right) dt = \int Q e^{-t(1+J)}\, dt$$

$$A_n(t) = e^{Jt} \int Q e^{-t(1+J)}\, dt = Q e^{Jt}\left(\frac{-1}{1+J} e^{-t(1+J)} + C_n\right)$$

Also: $A_n(0) = 0$ implies $0 = Q\left(\frac{-1}{1+J} + C_n\right)$ so $C_n = \frac{1}{1+J}$

Thus: $A_n(t) = \frac{Q}{1+J}\left(e^{Jt} - e^{-t}\right)$

$$\boxed{A_n(t) = \frac{2L^3\left((-1)^n-1\right)}{(Ln\pi)^2 - k(n\pi)^4}\left(e^{-k\left(\frac{n\pi}{L}\right)^2 t} - e^{-t}\right)}$$