LINEARITY

A **LINEAR OPERATOR** $\mathcal{L}$ satisfies $\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)$

for any functions u_1 and u_2, and constants c_1 and c_2.

examples: derivative, integral

HEAT OPERATOR: $\mathcal{L}(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}$

[Heat Equation: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ or $\mathcal{L}(u) = 0$]

Why is this linear?

$$\mathcal{L}(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2)$$

since derivative is a linear operator

$$= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1\frac{\partial u_1}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial u_2}{\partial t} - kc_2\frac{\partial^2 u_2}{\partial x^2}$$

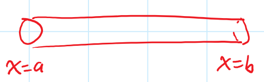
$$= c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)$$

LINEAR EQUATION: $\mathcal{L}(u(x,t)) = f(x,t)$, where $\mathcal{L}$ is a linear operator and $f(x,t)$ is known

LINEAR HOMOGENEOUS EQUATION: $\mathcal{L}(u(x,t)) = 0$

LINEAR BOUNDARY CONDITIONS: $\mathcal{L}(u(a,t)) = f_1(t)$ and $\mathcal{L}(u(b,t)) = f_2(t)$

EXAMPLES: $u(a,t) = 5$ or $\frac{\partial u}{\partial t}(b,t) = 0$

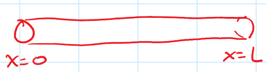


NON-EXAMPLE: $[u(0,t)]^2 = 1$

PRINCIPLE OF SUPERPOSITION: If u_1 and u_2 satisfy a linear homogeneous equation, then any linear combination $c_1u_1 + c_2u_2$ also satisfies the same linear homogeneous equation.

SEPARATION OF VARIABLES

PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$



Boundary Conditions: $u(0,t) = 0$, $u(L,t) = 0$

[Initial Condition: $u(x,0) = f(x)$ — ignore for now]

1. Look for a solution: $u(x,t) = \phi(x)\,G(t)$ $\Leftarrow$ ϕ and G are some unknown functions

Plug in to the PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

$$\phi(x)\frac{dG(t)}{dt} = k\frac{d^2\phi(x)}{dx^2}G(t)$$

Separate x from t:

2. (a)

$$\underbrace{\underbrace{\frac{1}{kG(t)}\frac{dG}{dt}}_{\text{function of } t \text{ only}} = \underbrace{\frac{1}{\phi(x)}\frac{d^2\phi}{dx^2}}_{\text{function of } x \text{ only}}}_{\text{These can only be equal if they are in fact constant!}} = -\lambda$$

where λ is some constant and the $-$ sign is for convenience later

We obtain: $\frac{1}{kG(t)}\cdot\frac{dG}{dt} = -\lambda$ and $\frac{1}{\phi(x)}\cdot\frac{d^2\phi}{dx^2} = -\lambda$ $\leftarrow$ two ODEs

(b) $\frac{\partial G}{\partial t} = -\lambda k G$ has solution $G(t) = Ae^{-\lambda k t}$

3. BVP: $\frac{d^2\phi}{dx^2} = -\lambda\phi$ with $\phi(0) = 0$, $\phi(L) = 0$

$\phi'' + \lambda\phi = 0$, $s^2 + \lambda = 0$

Boundary:
$u(0,t) = 0$
$\phi(0)\,G(t) = 0$
$\phi(0) = 0$ to avoid the trivial solution

$\lambda < 0$: two real roots: $r = \pm\sqrt{-\lambda}$

solution: $\phi(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

boundary: $\phi(0) = 0 = c_1 + c_2$ so $c_2 = -c_1$

$\phi(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$

$= c_1\underbrace{\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right)}_{\text{not zero}} = 0$ so $c_1 = 0 = c_2$

Only the trivial solution

$\lambda = 0$: $\phi'' = 0$, so the solution is $\phi(x) = ax + b$

boundary: $\phi(0) = b = 0$

$\phi(L) = aL + 0 = 0$ so $a = 0$

only the trivial solution

$\lambda > 0$: complex roots $r = \pm i\sqrt{\lambda}$

solution: $\phi(x) = c_1\sin(\sqrt{\lambda}\,x) + c_2\cos(\sqrt{\lambda}\,x)$

boundary: $\phi(0) = 0 \Rightarrow 0 = \cancel{c_1\sin(0)} + c_2\cos(0) \Rightarrow 0 = c_2$

$\phi(L) = 0 \Rightarrow \underline{0 = c_1\sin(\sqrt{\lambda}\,L)}$

$$\phi(L) = 0 \Rightarrow \underbrace{0 = c_1 \sin(\sqrt{\lambda}\, L)}$$

We want $c_1 \neq 0$ to obtain a nontrivial solution.

What does this imply about λ? To be continued...