Multi-D Heat Equation

Conservation eq: for an arbitrary region $R \subseteq \mathbb{R}^3$

rate of change of heat energy in R = heat energy flowing across boundaries of R (per unit time) + heat energy generated in R (per unit time)

$$\frac{d}{dt}\iiint_R c\rho u \, dV = -\oiint \phi \cdot \vec{n}\, dS + \iiint_R Q\, dV$$

u = temperature

ρ = mass density

c = specific heat

ϕ = heat flux vector

$\vec{n}$ = outward unit normal vector to the boundary of R

Q = heat energy produced in R per unit volume, unit time

negative sign: flow out of R decreases the total heat in R

By the divergence theorem, the middle term becomes $-\iiint_R \nabla\cdot\phi \, dV$.

$$\boxed{\oiint \vec{A}\cdot\vec{n}\, dS = \iiint_R \nabla\cdot\vec{A}\, dV}$$

Combine all 3 triple integrals: $\displaystyle\iiint_R \left[c\rho\frac{\partial u}{\partial t} + \nabla\cdot\phi - Q\right]dV = 0$

Since R is arbitrary, the integrand must be zero, so: $c\rho\frac{\partial u}{\partial t} = -\nabla\cdot\phi + Q$.

Fourier's Law: heat flux vector is proportional to the temperature gradient

$$\phi = -K_0 \nabla u \qquad \nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle$$

Thus: $c\rho\frac{\partial u}{\partial t} = -\nabla\cdot\left(-K_0 \nabla u\right) + Q$

$$c\rho\frac{\partial u}{\partial t} = K_0 \nabla\cdot\nabla u + Q$$

Simplify: let $k = \frac{K_0}{c\rho}$, assume $Q = 0$ (no sources)

$$\frac{\partial u}{\partial t} = k\,\nabla\cdot\nabla u$$

$$\boxed{\frac{\partial u}{\partial t} = k\,\nabla^2 u}$$

Laplacian: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Steady State: let $\frac{\partial u}{\partial t} = 0$, and the heat equation becomes:

$$\boxed{0 = \nabla^2 u}$$

Laplace's Equation

or, if we have a source:

$$\boxed{\nabla^2 u = \frac{-Q}{K_0}}$$

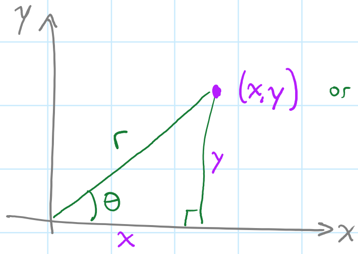
Poisson's Equation

$$\boxed{0 = \nabla^2 u}$$ or, if we have a source: $$\boxed{\nabla^2 u = \frac{\ \cdot\ }{K_0}}$$

Laplace's Equation Poisson's Equation

WORKSHEET: LAPLACE'S EQ. IN POLAR COORDS

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$\cos\theta = \frac{x}{r}$

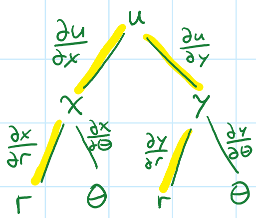
$\sin\theta = \frac{y}{r}$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

① $\frac{\partial x}{\partial r} = \cos\theta$ $\qquad \frac{\partial y}{\partial r} = \sin\theta$

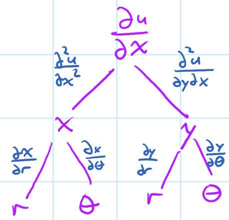
$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

② Multivariable chain rule



$u = u(x,y)$

$\frac{\partial u}{\partial x}$ is also a function of x and y, or of r and θ



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial x\partial y}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right)\sin\theta \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta
\end{aligned}$$

← multivariable chain rule

③ $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

PRODUCT RULE

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta) + \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)(r\cos\theta) + \frac{\partial u}{\partial y}(-r\cdot\sin\theta) \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial y}{\partial\theta}\right)(-r\sin\theta) - \frac{\partial u}{\partial x}(r\cos\theta) + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial\theta}\right)(r\cos\theta) - \frac{\partial u}{\partial y}r\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial x\partial y}(r\cos\theta)\right)(-r\sin\theta) - \frac{\partial u}{\partial x}(r\cos\theta) + \left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right)(r\cos\theta) - \frac{\partial u}{\partial y}r\sin\theta \\
&= \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta
\end{aligned}$$

(4) $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(\underbrace{\frac{\partial^2 u}{\partial x^2}\cos^2\theta} + \cancel{2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta} + \underbrace{\frac{\partial^2 u}{\partial y^2}\sin^2\theta}\right)$$

$$+ \left(\underbrace{\frac{\partial^2 u}{\partial x^2}\sin^2\theta} - \cancel{2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta} + \underbrace{\frac{\partial^2 u}{\partial y^2}\cos^2\theta} - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial u}{\partial r}$$

(5) Thus: $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$