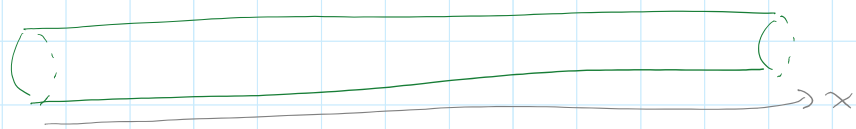
CONSERVATION EQUATION

rate of change of a quantity over time = amount flowing in across boundaries per unit time + amount generated inside per unit time

NOTE: "quantity" might be heat energy (temperature), a substance, pollution, traffic, chemical diffusing, etc.

1-D heat flow:

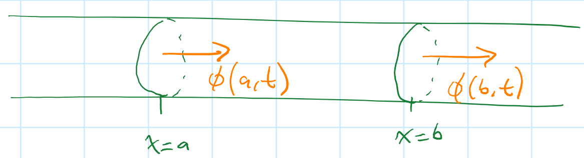


$u(x,t)$ = density of the quantity (per unit volume) at position x, time t

$\phi(x,t)$ = flux: flow (to the right) at position x, time t

$Q(x,t)$ = quantity generated per unit volume at position x, time t

Consider a segment $a < x < b$:



conservation eq:

$$\frac{d}{dt}\left(\text{total quantity}\right) = \left(\text{flow in}\right) + \left(\text{quantity generated}\right)$$

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)\,dx = \underbrace{\phi(a,t) - \phi(b,t)}_{\text{FTC}} + \int_a^b Q(x,t)\,dx}$$

integral form of conservation law

FTC: $\int_a^b f'(x)\,dx = f(b) - f(a)$

by continuity of u

$$\int_a^b \frac{\partial u(x,t)}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q(x,t)\,dx$$

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right) dx = 0$$

Since this holds for any a and b, we conclude:

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0$$

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unknown functions: u, ϕ

$$\boxed{\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q}$$

differential form of conservation law

usually, Q is known

TEMPERATURE: Fourier's law of heat conduction: $\phi = -K_0 \frac{\partial u}{\partial x}$

thermal conductivity constant

Conservation law becomes:

$$\underbrace{c\rho \frac{\partial u}{\partial t}}_{} = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q$$

thermal energy = (specific heat c)(mass density ρ)(temperature u)

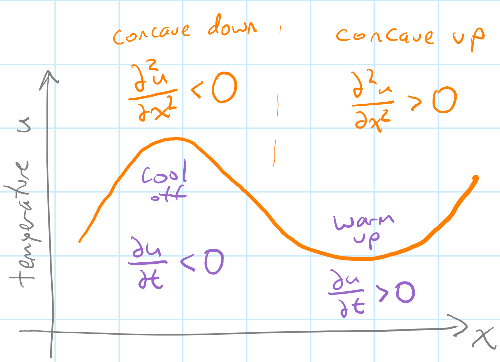
$$\Rightarrow \quad c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

Let "thermal diffusivity" $k = \frac{K_0}{c\rho}$. Also, assume $Q = 0$ (no sources).

Then: $$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$ **HEAT EQUATION**

Intuition for the heat equation:

(change in temperature over time) is proportional to concavity of graph of $u(x)$



Time derivative is proportional to the second spatial derivative.

WORKSHEET: CONSERVATION LAWS

1. (a) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

   (b) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = 0$

   (c) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ "inviscid" = no viscosity (frictionless flow)

(d) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = 0$

(e) $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = u(1-u)$

2. (a) total heat energy flowing out lateral surface per unit time: $\int_0^L w(x,t)\,P(x)\,dx$

(b) conservation law:

$$\frac{d}{dt}\int_0^L c(x)\,\rho(x)\,u(x,t)\,A\,dx = \phi(0,t)A - \phi(L,t)A - \int_0^L w(x,t)\,P(x)\,dx$$

$$= -K_0 u_x(0,t)A + K_0 u_x(L,t)A - \int_0^L w(x,t)\,P(x)\,dx$$

(c) Arbitrary bounds $a < b$:

$w(x,t) = h(x)\big(u(x,t) - \gamma(x,t)\big)$

$$\frac{d}{dt}\int_a^b c\rho u A\,dx = \int_a^b \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)A\,dx - \int_a^b h(u-\gamma)P\,dx$$

Combine integrals and divide by A:

$$\int_a^b \left(c\rho\frac{\partial u}{\partial t} - K_0\frac{\partial^2 u}{\partial x^2} + \frac{P}{A}h(u-\gamma)\right)dx = 0$$ for arbitrary bounds $a < b$.

Thus: $c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} - \frac{P}{A}(u-\gamma)h$

(d) c, ρ, K_0 constant, $A = \pi r^2$, $P = 2\pi r$, $\gamma = 0$

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} - \frac{2h}{r}u$$

(e) If $u(x,t) = u(t)$ and $u(0) = u_0$:

$c\rho\frac{\partial u}{\partial t} = 0 - \frac{2h}{r}u(t)$, so $\frac{\partial u}{\partial t} = \frac{-2h}{c\rho r}u(t)$, which has solution $\boxed{u(t) = u_0 e^{\frac{-2h}{c\rho r}t}}$