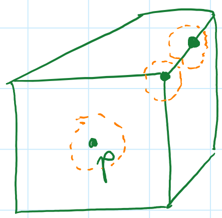
From last time: DEFINING "POLYHEDRON"

A polyhedron is composed of polygonal faces and satisfies the following:

1. **INTERSECTION CONDITION:** Any two faces may only intersect at a single vertex or along a single edge.

2. **LOCAL TOPOLOGY:** Every point p on the surface of a polyhedron P has a neighborhood homeomorphic to an open disk.



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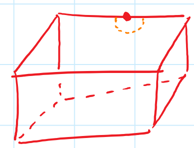
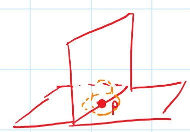
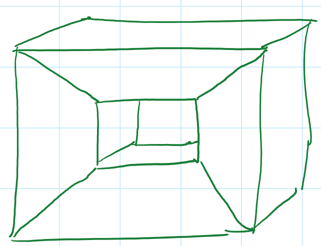


open box (no top)



3. **GLOBAL TOPOLOGY:** Surface of the polyhedron is connected. (Tunnels are OK, but floating cavities are not OK.)



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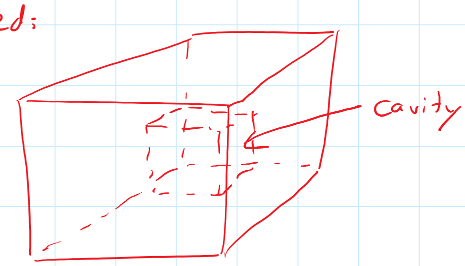
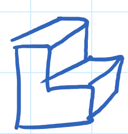


PROBLEM: Compute $V - E + F$ for some nearby polyhedron.

num. vertices, edges, faces

Fig. 6.10 (right): -12

Dodecahedron: 2



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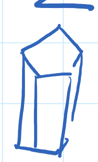

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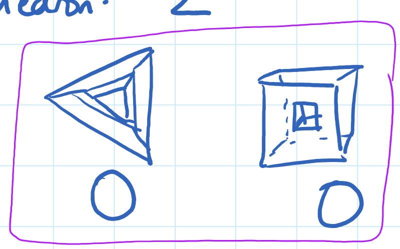

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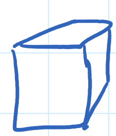

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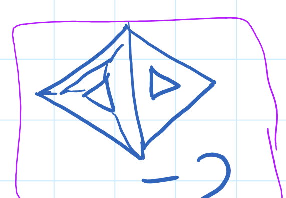


-2

Great or Small Rhombicuboctahedron: 2

homeomorphic to a sphere

Observe: If polyhedron P has no holes/tunnels, then $\chi(P) = V - E + F = 2$. ← Euler's Polyhedron Formula

EULER CHARACTERISTIC

For each hole/tunnel, $V-E+F$ decreases by 2.

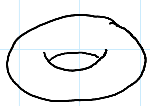
$$\chi(P) = V-E-F = 2-2g,$$

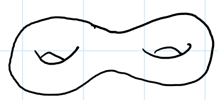
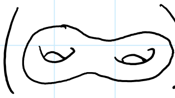
where g is the GENUS of the polyhedron.

GENUS: number of holes or tunnels in the polyhedron

This applies to smooth surfaces, too:

sphere:  genus 0, $\chi($  $) = 2$

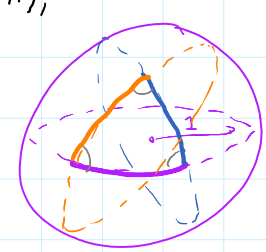
torus:  genus 1 $\chi($  $) = 0$

double torus:  genus 2 $\chi($  $) = -2$

Sketch of proof that $\chi(P) = 2$ if P is a convex polyhedron.

FACT: On a sphere of radius 1 (so surface area 4π), any triangle defined by great circle arcs satisfies:

angle sum = area + π



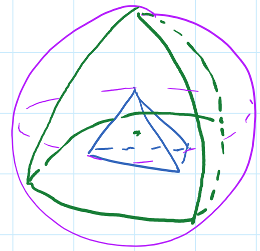
Polyhedron P is convex.

Triangulate the surface of P. (This doesn't change $V-E+F$)

Imagine P is inside the sphere of radius 1.

Project P onto the sphere (imagine a light shining from within P).

Add up all angles of all triangles on the sphere: $(4+F)\pi$



$$\sum_{\text{faces } F} \left(\text{area}(F)+\pi\right) = \text{Surface area of sphere} + \pi\left(\text{num. faces}\right)$$
$$= 4\pi + F\pi$$

At each vertex, the angles sum to 2π, so sum of all angles on the sphere is $2\pi V$

Thus: $(4+F)\pi = 2\pi V$, so $4+F = 2V$

Count edges in two different ways: $3F = 2E$ so $F = 2E - 2F$

So: $4 + (2E - 2F) = 2V$

$2 + E - F = V$ and thus $\boxed{2 = V - E + F}$

PROBLEM: Let P be a polyhedron of genus 0. If every face of P is either a pentagon or a hexagon, and if the degree of each vertex is 3, then how many faces are pentagons?

Suppose there are n pentagons and m hexagons.

Then: $F = n + m$

Total degree: $5n + 6m = 3V = 2E$

so: $V = \frac{5n+6m}{3}$ and $E = \frac{5n+6m}{2}$

Euler characteristic: $V - E + F = 2$

$$6\left(\frac{5n+6m}{3} - \frac{5n+6m}{2} + (n+m)\right) = 2\cdot 6$$

$$10n + \cancel{12m} - 15n - \cancel{18m} + 6n + \cancel{6m} = 12$$

$$\boxed{n = 12}$$