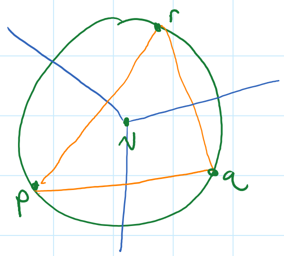
The dual graph of the Voronoi diagram of a point set S is the Delaunay triangulation of S.

Theorem: A point v is a Voronoi vertex iff there is a circle centered at v with 3 (or more) sites on its boundary and none in its interior.

proof: $\Rightarrow$ If v is a Voronoi vertex, then it is incident to at least 3 Voronoi edges and at least 3 Voronoi cells, say $\underbrace{\mathrm{Vor}(p), \mathrm{Vor}(q), \mathrm{Vor}(r)}_{\text{voronoi regions, } p,q,r \text{ – sites}}$.



Then v is equidistant from p, q, and r.

If there were another site closer to v, then these Voronoi regions would not meet at v.

$\Leftarrow$ If a circle centered at v has sites p, q, r on its boundary and no sites in its interior, then the regions $\mathrm{Vor}(p)$, $\mathrm{Vor}(q)$, and $\mathrm{Vor}(r)$ must meet at v.

Theorem: If S is a point set with no four points cocircular, then the dual graph of $\mathrm{Vor}(S)$ is the Delaunay triangulation of S.

proof: By the previous theorem, the circumcircle of each triangle in the dual graph of $\mathrm{Vor}(S)$ has no sites in its interior.

Since no four points are cocircular, such a circle has exactly 3 sites on its boundary.

Thus, the dual graph of $\mathrm{Vor}(S)$ is a triangulation with the "empty circle property" from Chapter 3, which uniquely characterizes the Delaunay triangulation.

AN INCREMENTAL ALGORITHM

Given a Voronoi diagram (n sites) and a new site p.

Update the Voronoi diagram:

(1) Find the Voronoi region, say $Vor(q)$, that contains p. $O(n)$

(2) Compute the line segment $\overline{ab}$ that is the $\perp$ bisector to $\overline{pq}$, such that a and b are on the boundary of $Vor(q)$. $O(n)$

(3) Contine from a, constructing $Vor(p)$ segment by segment, until we get back to b. $O(n)$

(4) Remove the subdiagram inside the new region $Vor(p)$. $O(n)$?

n times

$O(n^2)$ algorithm