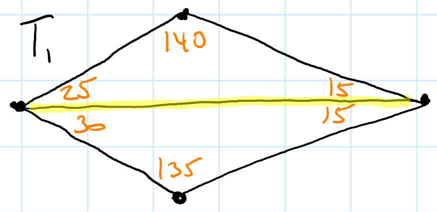
DELAUNAY TRIANGULATION

Triangulation T_1 is "fatter" than triangulation T_2 if the sorted list of angles of T_1 is lexicographically larger than that of T_2.

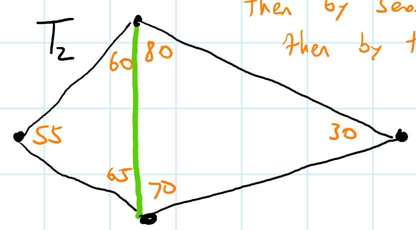
sort by first element, then by second, then by third, ...

example:

T_1



T_2



angles: 15, 15, 25, 30, 135, 140 | 30, 55, 60, 65, 70, 80

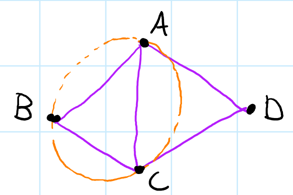
Here, T_2 is "fatter" than T_1.

The Delaunay triangulation of a point set S is the fattest triangulation of S.

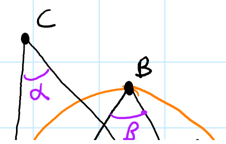
Equivalently, the Delaunay triangulation has only "legal" edges, where an edge is illegal if flipping it results in a fatter triangulation.

EXPLORATION: Justify the following:

Let $\overline{AC}$ be an edge of a triangulation with triangles ABC and ACD. Then $\overline{AC}$ is a legal edge of the Delaunay triangulation if and only if D is outside of the circumcircle of $\triangle ABC$.



Useful theorem (Thales' Theorem):

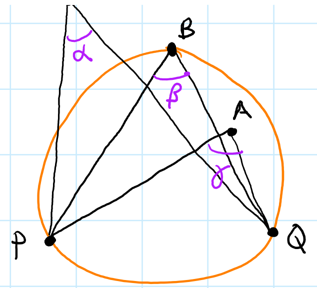


Useful theorem (Thales' Theorem):

For points P, Q, A, B, C as shown,

$\measuredangle PAQ > \measuredangle PBQ > \measuredangle PCQ$

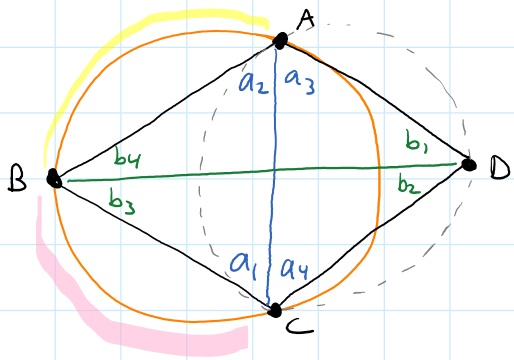
$\gamma > \beta > \alpha$



proof: Suppose D is outside the circumcircle of $\triangle ABC$

By Thales' Theorem: $a_1 > b_1$

$a_2 > b_2$, $a_3 > b_3$, $a_4 > b_4$



Triangulation containing $\overline{AC}$ has angles:

$a_1, a_2, a_3, a_4, b_1+b_2, b_3+b_4$

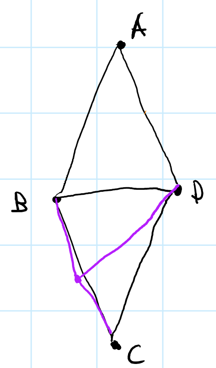
Triangulation containing $\overline{BD}$ has angles:

$b_1, b_2, b_3, b_4, a_1+a_4, a_2+a_3$

smallest

The smallest angle is one of $\{b_1, b_2, b_3, b_4\}$, so the triangulation containing $\overline{AC}$ is fatter, and thus is the Delaunay triangulation.

The other direction is similar: if D is inside the circle, then the Delaunay triangulation contains $\overline{BD}$, not $\overline{AC}$.



GEOMETRIC CHARACTERIZATION OF THE DELAUNAY TRIANGULATION:

T is a Delaunay Triangulation of a point set S if and only if no point of S lies inside the circumcircle of any triangle in T.