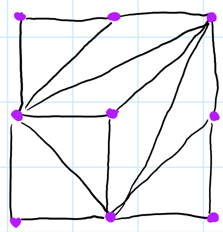
TRIANGULATIONS OF POINTS IN THE PLANE

A triangulation of a planar point set S is a subdivision of the plane by a maximal set of noncrossing edges whose vertex set is S.

can't add more edges

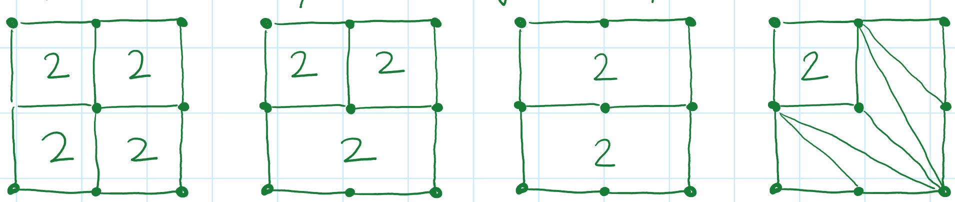
example: S



triangulation

Warm-Up: For the 9-point set S above, how many different triangulations are there? How many triangles are there in each triangulation?

Count by which horizontal/vertical edges are present



16 triangulations

8 triangulations
× 4 rotations
32

4 triangulations
× 2 rotations
8

2 triangulations
× 4 rotations
8

64 total triangulations, each with 8 triangles

Number of triangulations grows fast!

4x4 grid of dots: 46,456 triangulations

5x5 grid of dots: 736,983,568 "

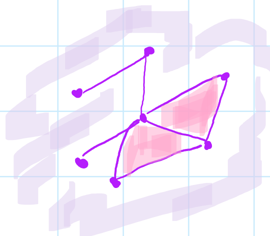
THEOREM: Let S have h points on its convex hull and k points inside the hull (not all points collinear). Then each triangulation of S has $2k+h-2$ triangles.

In our example, $h=8$, $k=1$, so $2k+h-2 = 2(1)+8-2 = 8$

EULER'S FORMULA: Let G be a connected planar graph with V vertices, E edges, and F faces. Then:

$$V - E + F = 2.$$

Example:



$V=7$
$E=8$
$F=3$

$V-E+F = 7-8+3 = \underline{2}$

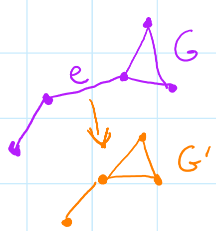
proof that $V-E+F=2$ for any connected planar graph:

Induction on the number of edges E:

base case: If $E=0$, then there is only one vertex: $V=1$ and $F=1$. Then: $V-E+F = 1-0+1 = 2$

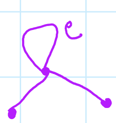
induction: Assume the formula holds for all graphs with $E-1$ edges. (E integer >0)

Choose any edge e of graph G and remove it:



- If e connects two vertices, then contract e, identifying its endpoints as a single vertex. This reduces both E and V by 1.

$$\text{So: } (V-1)-(E-1)+F = 2$$
$$V-E+F = 2.$$



- If e is a loop, then delete e, reducing E and F by 1.

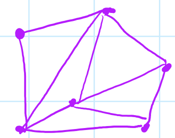
$$\text{So: } V-(E-1)+(F-1) = 2$$
$$V-E+F = 2.$$

Proof of theorem:

Let $n = h + k$, and t be the number of triangles.

Then there are $t+1$ faces, including the outside face.

- t triangles have $3t$ total edges.
- outside face has h edges.



So $3t + h$ counts every edge twice;

thus there are $\frac{3t+h}{2}$ edges.

Euler's formula: $V - E + F = n - \frac{3t+h}{2} + (t+1) = 2$

$$2n - 3t - h + 2t + 2 = 4$$

$$2(k+h) - t - h = 2$$

$$\boxed{2k + h - 2 = t}$$

We could also solve for the number of edges:

$$E = \frac{3t+h}{2} = -2 + (t+1) + n$$

$$= -1 + t + n = -1 + (2k+h-2) + (k+h)$$

$$\boxed{E = 3k + 2h - 3}$$