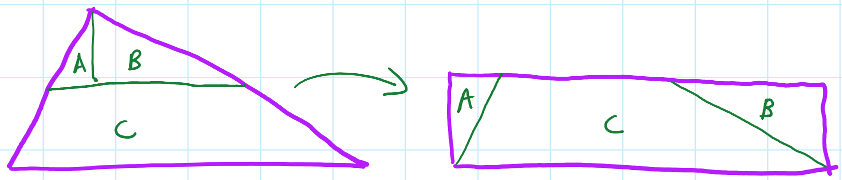
DISSECTION of a polygon P cuts P into a finite number of smaller polygons

Polygons P and Q are **SCISSORS CONGRUENT** if P can be dissected into $P_1, \ldots, P_n$, which can be reassembled by translations and rotations to form Q.

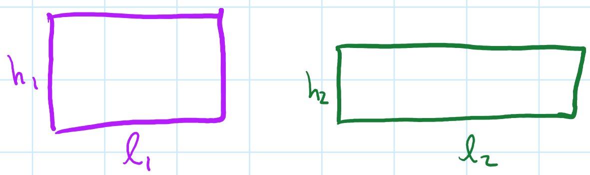
slide

Note: not reflections

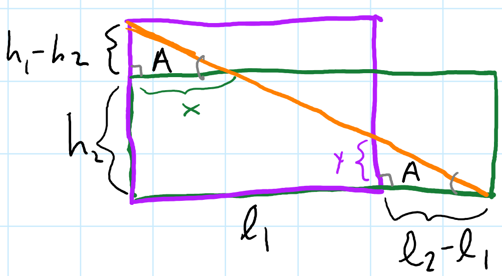
LEMMA 1.49: Every triangle is scissors congruent to some rectangle



LEMMA 1.50: Any two rectangles of the same area are scissors congruent.



Assume $h_2 < h_1 \leq l_1 < l_2 < 2l_1$



Observe: $\dfrac{h_1 - h_2}{l_2 - l_1} = \dfrac{h_1}{l_2}$

$$l_2 h_1 - \cancel{l_2 h_2} = h_1 l_2 - \cancel{h_1 l_1}$$

By similar triangles,

$$\frac{h_1 - h_2}{x} = \frac{y}{l_2 - l_1} = \frac{h_1}{l_2} = \frac{h_1 - h_2}{l_2 - l_1}$$

So $x = l_2 - l_1$

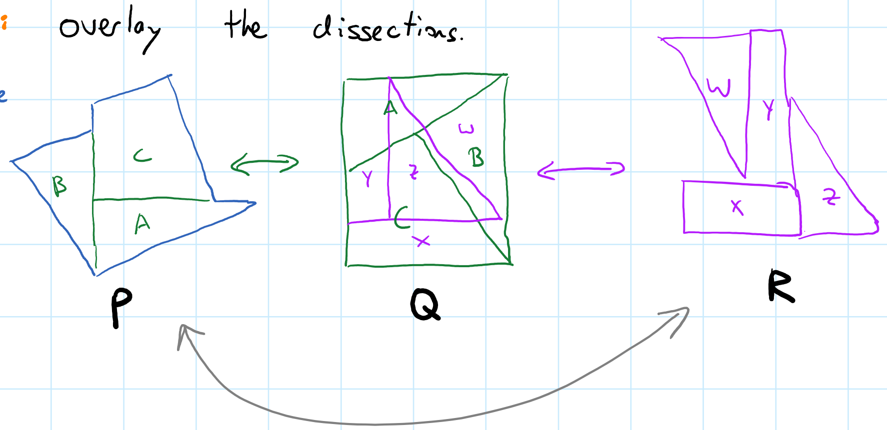
so $y = h_1 - h_2$

LEMMA: Scissors congruence is transitive.

If P is scissors congruent to Q, and Q is scissors congruent to R, then P is scissors congruent to R.

proof: overlay the dissections.

example



THEOREM 1.53 (BOLYAI-GERWEIN):

Any two polygons of the same area are scissors congruent.

https://dmsm.github.io/scissors-congruence/

Does a similar theorem hold in 3D?

3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes.

Hilbert's 3rd problem (1900)

In two letters to Gerling, Gauss * expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, *i. e.*, in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved.† Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in *specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.*‡

1903: Dehn showed answer is no

example: cube and a regular tetrahedron are not scissors congruent

key: Dehn invariant — a function on polyhedra which is unchanged by dissection and rearrangement

section 1.5

1965: Sydler showed that if P and Q have the same volume and Dehn invariant, then they are scissors congruent