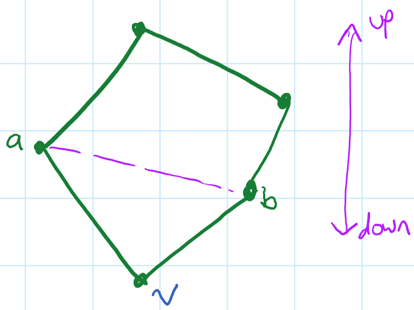
Proof Techniques: line sweep, induction

LEMMA 1.3: Every polygon P with more than 3 vertices has a diagonal.

little theorem, used for proving something else



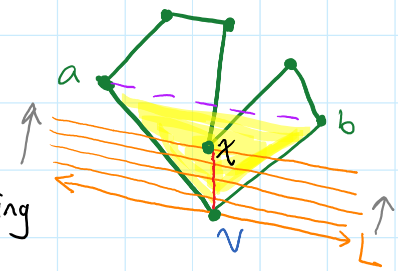
Proof: Let v be the lowest vertex of P.

(rightmost, if there are multiple lowest vertices)

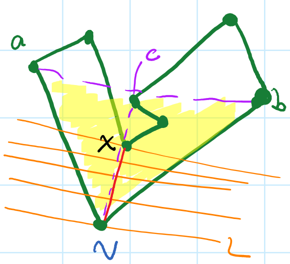
Let a, b be the neighboring vertices of v.

If the segment $\overline{ab}$ is a diagonal, then done.

Otherwise, then the triangle avb contains at least one vertex.



Let L be the line parallel to $\overline{ab}$, passing through v.



Sweep L upward toward ab. Let x be the first vertex that L meets along this sweep.

Then the segment $\overline{vx}$ is within P, and is thus a diagonal.

THEOREM 1.4: Every polygon has a triangulation.

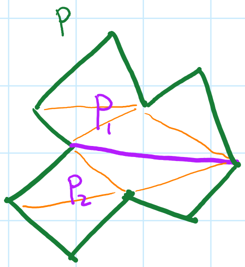
Proof: by induction on the number of vertices n of polygon P.

Base Case: If $n=3$, then P is a triangle. Thus, P has a triangulation.

Induction: Let $n>3$, and assume that every polygon with fewer than n vertices has a triangulation.

We will show that polygons with n vertices have triangulations.

By Lemma 1.3, P has a diagonal, which cuts P into two polygons P_1 and P_2.

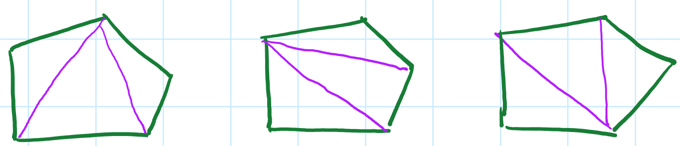


Then each of P_1 and P_2 has fewer than n vertices.
By the assumption, P_1 and P_2 have triangulations.
Combine the triangulations* of P_1 and P_2 to obtain a triangulation of P.

* P_1 and P_2 are disjoint by the Jordan Curve Theorem

THEOREM 1.8: If P has n vertices, then every triangulation of P has $n-2$ triangles and $n-3$ diagonals.

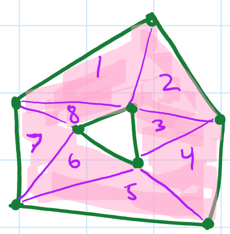
proof: in book, by induction.



3 triangles
2 diagonals

$n=5$ vertices

Polygon with holes:



$n=8$ vertices
$h=1$ hole
triangles: 8

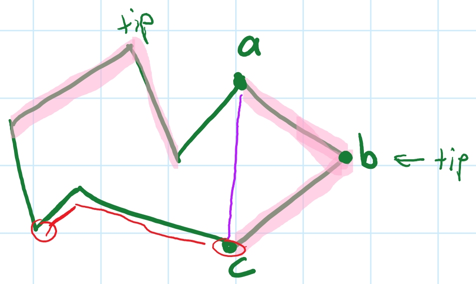
Question: How does the number of triangles in a triangulation depend on h and n?

$$n-2+2h = \text{num. triangles}$$

Number of triangulations: A polygon P with n vertices has between 1 an C_{n-2} triangulations.

Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\cdot\frac{(2n)!}{n!\,n!}$

"Ear" of a polygon: If a, b, c are consecutive vertices and $\overline{ac}$ is a diagonal, then abc is called an ear of the polygon.



Cor. 1.9: Every polygon with more than 3 vertices has at least two ears with non-adjacent tips.