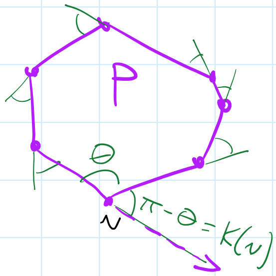
CURVATURE OF POLYGONS



angular defect: measure of how much the polygon bends at each vertex

$K(v) = \pi - \text{interior angle at vertex } v$

add up the angular defect of all vertices: you get 2π

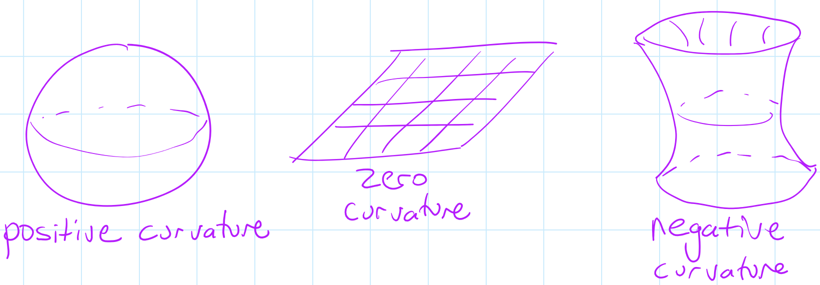
$$\sum_{v \in P} K(v) = 2\pi$$

[The sum of the interior angles of a n-gon:

$n\pi - 2\pi = (n-2)\pi$]

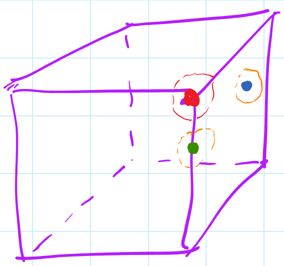
CURVATURE IN 3D

surfaces:



POLYHEDRA: The curvature $K(p)$ at a point p on a polyhedral surface is 2π minus the sum of the face angles at p.

EXAMPLES:

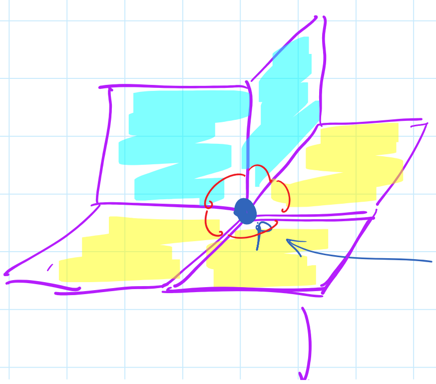


point on a face: $K(p) = 2\pi - 2\pi = 0$

point on an edge: $K(p) = 2\pi - (\pi + \pi) = 0$

point on a vertex: $K(p) = 2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}$

All of the curvature is concentrated at the vertices.

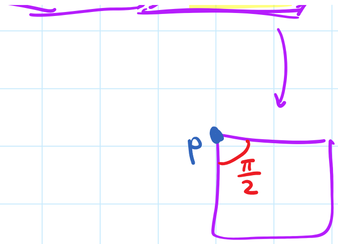


Curvature at p:

$K(p) = 2\pi - \left(5 \cdot \frac{\pi}{2}\right) = -\frac{\pi}{2}$

this polyhedron is negatively curved at p

curved at p

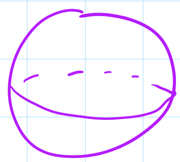


POLYHEDRAL GAUSS-BONNET THEOREM

For a polyhedron P, the total curvature satisfies

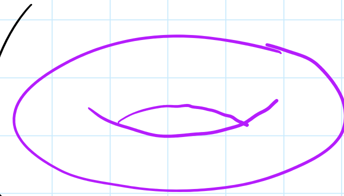
$$\underbrace{\sum_{v \in P} K(v)}_{\text{sum of curvature at vertices}} = 2\pi \underbrace{\chi(P)}_{\text{Euler characteristic}}$$

SMOOTH SURFACES:

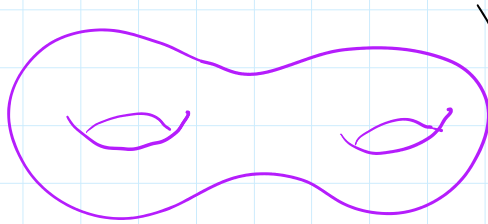
$\chi\Big($  $\Big) = 2$

genus zero

torus

$\chi\Big($  $\Big) = 0$

genus 1

$\chi\Big($  $\Big) = -2$

genus 2

For a surface S of genus g, $\chi(S) = 2-2g$.

Euler characteristic.

In differential geometry, the Gaussian curvature is defined for points on a surface.

GAUSS-BONNET THEOREM:

For a surface S without boundary,

$$\int_S K \, dA = 2\pi \underbrace{\chi(S)}_{\text{Euler characteristic}}.$$

surface integral

curvature at a point

Polyhedral Curvature

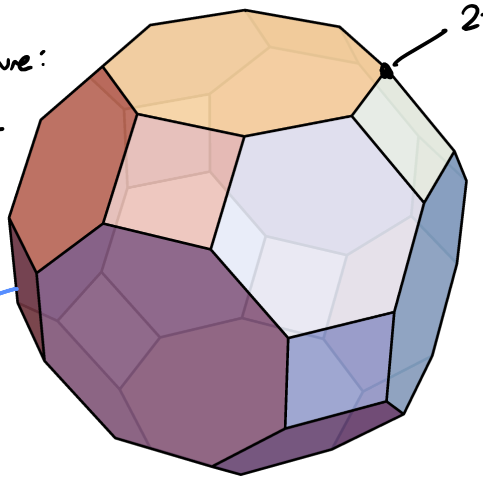
Math 282 Computational Geometry

Compute the Gaussian curvature at each vertex of the polyhedron. Then add up the Gaussian curvature over all vertices of the polyhedron.

Total curvature:

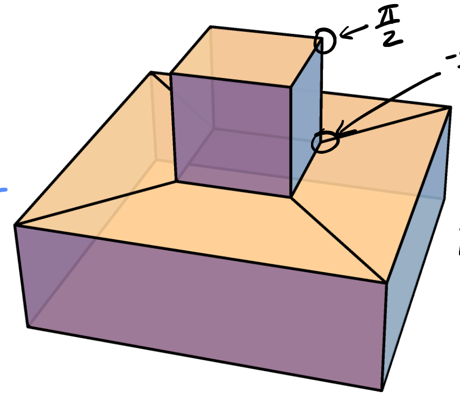
$48\left(\frac{\pi}{12}\right) = 4\pi$

$\chi(P) = 2$



$2\pi - \left(\frac{\pi}{2} + \frac{3\pi}{4} + \frac{2\pi}{3}\right) = \frac{\pi}{12}$

2



$\frac{\pi}{2}$

$-\frac{\pi}{2}$

Total curvature:

$12\left(\frac{\pi}{2}\right) + 4\left(-\frac{\pi}{2}\right)$

$= 4\pi$

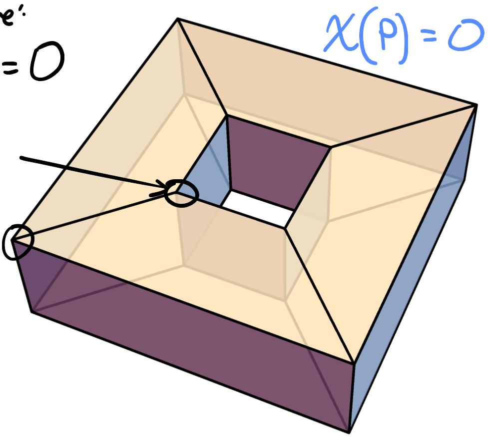
Total curvature:

$8\left(\frac{\pi}{2}\right) + 8\left(-\frac{\pi}{2}\right) = 0$

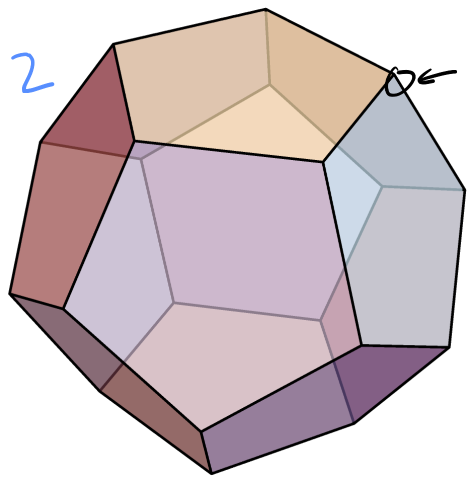
$\chi(P) = 0$

$-\frac{\pi}{2}$

$\frac{\pi}{2}$



2

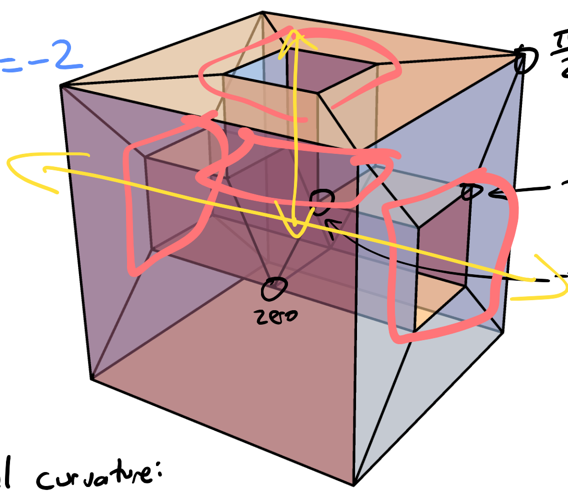


$2\pi - 3\left(\frac{3\pi}{5}\right) = \frac{\pi}{5}$

Total curvature:

$20\left(\frac{\pi}{5}\right) = 4\pi$

$\chi(P) = -2$



$\frac{\pi}{2}$

$-\frac{\pi}{2}$

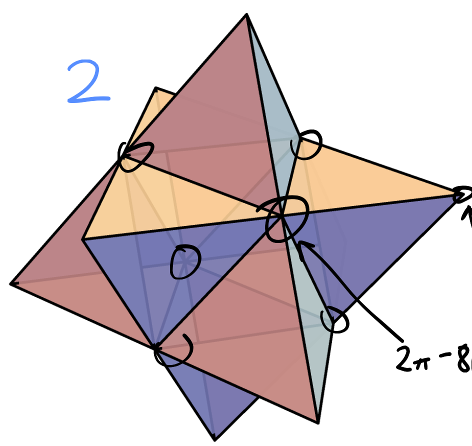
$-\frac{\pi}{2}$

zero

Total curvature:

$8\left(\frac{\pi}{2}\right) + 16\left(-\frac{\pi}{2}\right) = -4\pi$

2



π

$2\pi - 8\left(\frac{\pi}{3}\right) = -\frac{2\pi}{3}$

Total curvature:

$8(\pi) + 6\left(-\frac{2\pi}{3}\right) = 4\pi$

For each of the following polyhedra, let R be the region above the curve indicated by arrows. Verify the formula:

$$\sum_{v \in R \setminus \partial R} K(v) + \sum_{v \in \partial R} K(v) = 2\pi\chi(R).$$