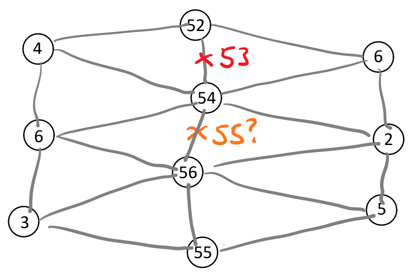
Which triangulation is best?

What does this mean?
Most efficient?
Shortest edge lengths?
Best for some application?

EXAMPLE:



Different triangulations
⇓
Different interpolations

LEXICOGRAPHICAL ORDER:

Generalization of dictionary order:

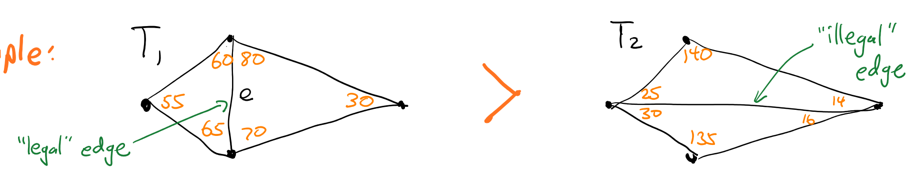
EXAMPLE: math < matrix < tetrahedron < triangle

Given two sequences $A = (a_1, a_2, a_3, \ldots)$ and $B = (b_1, b_2, b_3, \ldots)$ then we say $A < B$ if $a_i < b_i$ where i is the first index at which the sequences differ.

DELAUNAY TRIANGULATION

Let S be a planar point set with triangulations T_1 and T_2. We say T_1 is "fatter" than T_2 if the sorted list of angles in T_1 is lexicographically larger than that of T_2. We write this as $T_1 > T_2$.

example:



angles: (30, 55, 60, 65, 70, 80)

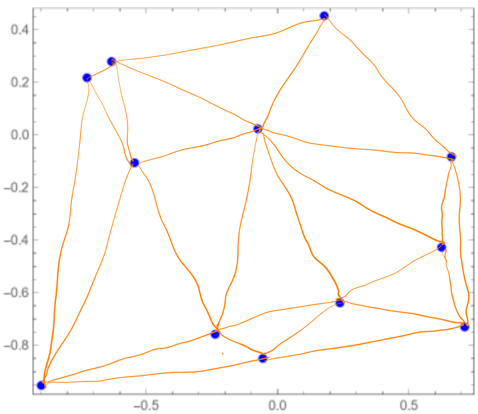
angles: (14, 16, 25, 30, 135, 140)

DEFINITION: Let e be an edge of triangulation T_1 and let Q be the quadrilateral in T_1 formed by the two triangles containing edge e. If Q is convex, let T_2 be the triangulation resulting from flipping edge e. Then e is a **legal edge** if $T_1 \geq T_2$ and e is an **illegal edge** if $T_1 < T_2$.

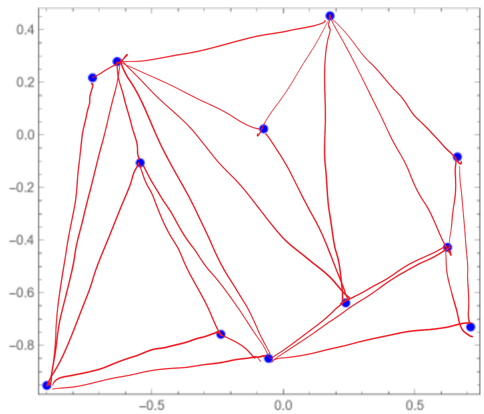
The **Delaunay triangulation** is the triangulation consisting of no illegal edges.

(The Delaunay triangulation is the "fattest" triangulation.")

Example:

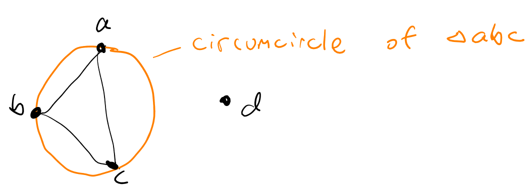


Delaunay Triangulation



Incremental Alg.

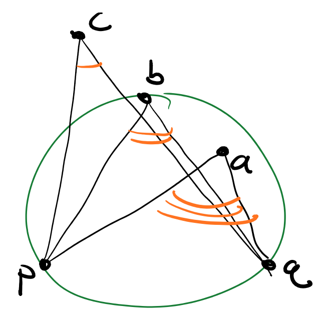
WORKSHEET:

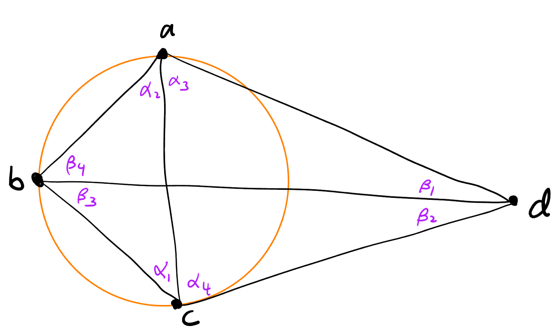


THALES'S THEOREM may be helpful:

Let p, q, b be points on a circle, a inside and c outside.

Then: $\measuredangle paq > \measuredangle pbq > \measuredangle pcq$





Suppose d is outside circe abc:

Proof of 1: By Thales's theorem, $\beta_1 < \alpha_1$ and $\beta_2 < \alpha_2$.

$\Leftarrow$ similarly, use circle acd to see $\beta_3 < \alpha_3$ and $\beta_4 < \alpha_4$,

Triangulation with $\overline{ac}$ has angles: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1+\beta_2, \beta_3+\beta_4$

Triangulation with $\overline{bd}$ has angles: $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_1+\alpha_2, \alpha_3+\alpha_4$

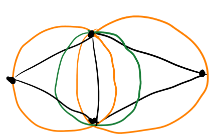
Since one of the β_i is smallest of all the angles shown, $\overline{ac}$ is a legal edge.

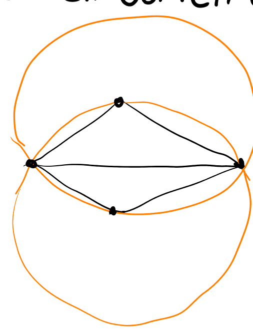
$\Rightarrow$ Similar reasoning.

EMPTY CIRCLE PROPERTY

T is a Delaunay triangulation if and only if no point of S lies in the circumcircle of any triangle in T.

Examples:





[For Delaunay triangulations, "general position" means no 4 points are cocircular.]

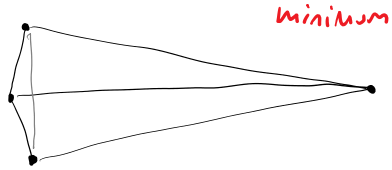
Also: an edge is legal if and only if there is some circle that goes through the endpoints of the edge but contains no points of S in its interior.

MINIMUM WEIGHT TRIANGULATION: The triangulation of S with the smallest total edge length.

Question: Is the Delaunay triangulation minimum weight?

No. Delaunay triangulation might not have minimum weight

Counterexample:

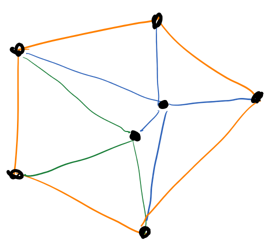


Finding the minimum weight triangulation is <u>NP-Hard.</u>

there is no polynomial-time algorithm $O(n^k)$ known

How could we compute the Delaunay triangulation?

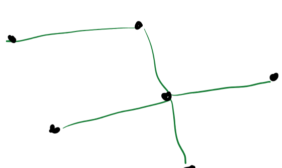
1. Add points from left to right, update triangulation for each point.
2. Find convex hull, add interior vertices one at a time.

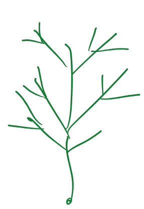


Bowyer-Watson algorithm (1981)

3. Focus on a single point, use circles to find Delaunay triangles incident to that point. Repeat for other points.
4. Pick a random triangulation. Check each convex quadrilateral an flip edges until you obtain the Delaunay triangulation.

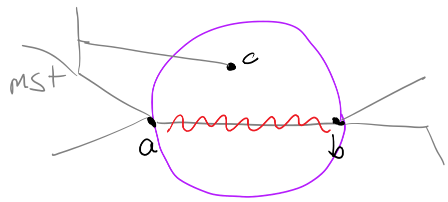
MINIMUM SPANNING TREE: shortest length tree (graph with no cycles) that spans all of the points





THEOREM: For a planar point set S, the minimum spanning tree of S is contained in the Delaunay triangulation of S.

proof by contradiction: Assume edge $\overline{ab}$ is in the min. spanning tree but not in the Delaunay triangulation.



Consider circle with diameter $\overline{ab}$.

Since edge $\overline{ab}$ is not legal, there must exist some point c in the circle.

Deleting edge $\overline{ab}$ from the min. spanning tree produces two disconnected sub-trees, one of which must include point c.

Then replacing edge $\overline{ab}$ with either $\overline{ac}$ or $\overline{bc}$ produces a smaller spanning tree, which is a contradiction.

Thus, all edges of min. spaning tree are part of the Delaunay triangulation.