Scissors Congruence

A **dissection** of a polygon P cuts P into a finite number of smaller polygons.

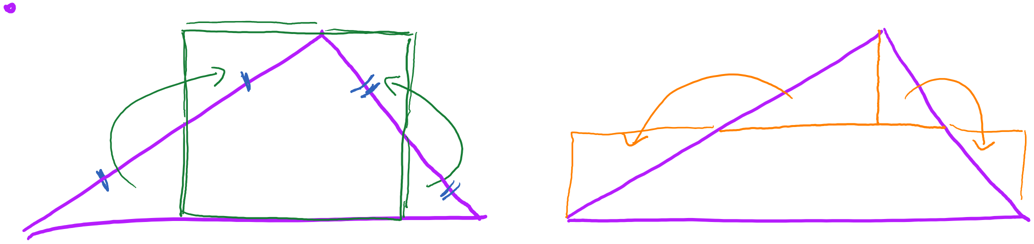
Polygons P and Q are **scissors congruent** if P can be dissected into $P_1, P_2, \ldots, P_n$, which can be reassembled by translations and rotations to form Q.

slide

NOTE: not reflections (flips)

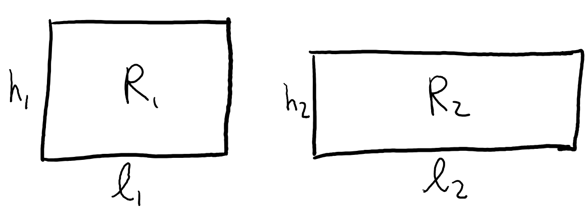
Question 1: Is every triangle is scissors congruent with some rectangle? *Find a proof or a counterexample.*

Yes!



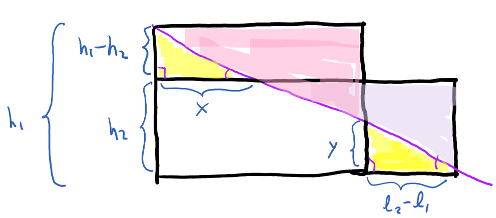
Question 2: Are every two rectangles of the same area are scissors congruent? *Find a proof or a counterexample.*

Yes!



Assume: $h_2 < h_1 \le \ell_1 < \ell_2 < 2\ell_1$

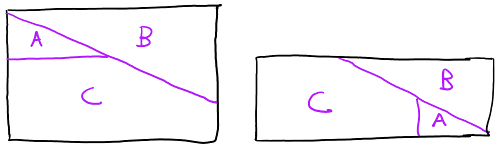
(If ℓ_2 is too big, the cut and stack R_2.)



By similar triangles,

$$\frac{h_1}{\ell_2} = \frac{h_1 - h_2}{x} = \frac{y}{\ell_2 - \ell_1}$$

Observe: $\frac{h_1 - h_2}{\ell_2 - \ell_1} = \frac{h_1}{\ell_2}$ why? cross-multiply



So $x = \ell_2 - \ell_1$
and $y = h_1 - h_2$

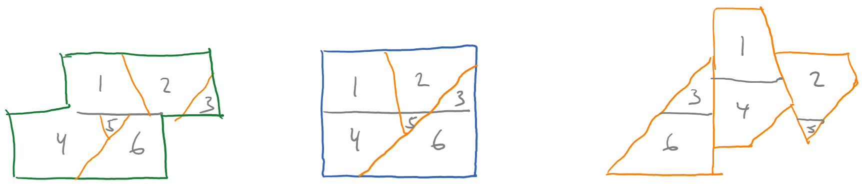
$\ell_2(h_1-h_2) = h_1(\ell_2-\ell_1)$
$\cancel{h_1\ell_2} - h_2\ell_2 = \cancel{h_1\ell_2} - h_1\ell_1$
$h_1\ell_1 = h_2\ell_2$ since both rectangles have same area.

Transitivity of Scissors Congruence

Question 3: Suppose polygon P is scissors congruent to polygon Q, and Q is scissors congruent to polygon R. Is P scissors congruent to R? *Find a proof or a counterexample.*

Yes: overlay the dissections of Q

example:



P Q R

Sketch of proof: Dissect P into $P_1, P_2, \ldots, P_n$ that rearrange to form Q.

Dissect R into $R_1, R_2, \ldots, R_m$ that rearrange to form Q

Let E be the set of all edges of the dissection of Q into $P_1, \ldots, P_n$ and let F be the set of all edges of the dissection of Q into $R_1, \ldots, R_m$.

Let $U = E \cup F$. Cut along all edges in U to dissect Q into polygons that can be rearranged to form either P or R.

Question 4: Let P and Q be polygons of the same area. Are P and Q scissors congruent? *Find a proof or a counterexample.*

Yes! Wallace - Bolyai - Gerwein theorem

proof: Let P and Q be polygons of the same area.

Triangulate P. Each triangle is scissors congruent to a rectangle of width 1. Stack these rectangles to form a rectangle of area equal to area of P.

By the same reasoning, Q is scissors congruent to that rectangle.

By transitivity of scissors congruence, P and Q are scissors congruent.

Does a similar theorem hold in 3D?

1900

3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes.

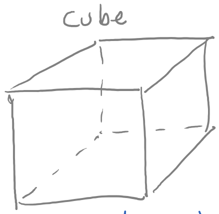
In two letters to Gerling, Gauss * expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, *i. e.*, in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved.† Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in *specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.*‡

Hilbert wants an example of two tetrahedra of the same volume that are not scissors congruent.

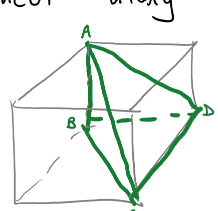
We we want to find an invariant: something that can be computed from polyhedra such that if two polyhedra are scissors congruent, then they have the same invariant.

DIHEDRAL ANGLE: angle between two faces of a polyhedron that meet along an edge.

cube



all dihedral angles are $\frac{\pi}{2}$



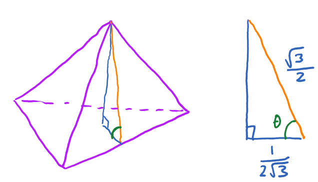
along BC: $\frac{\pi}{2}$

along AD: $\frac{\pi}{2}$

along CD: $\frac{\pi}{4}$

along AC: $\frac{\pi}{3}$

regular tetrahedron



$$\cos\theta = \frac{\frac{1}{2\sqrt{3}}}{\frac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}\sqrt{3}}\cdot\frac{\cancel{2}}{\sqrt{3}} = \frac{1}{3}$$

$$\theta = \arccos\left(\tfrac{1}{3}\right) \approx 70.53^\circ$$

not a rational multiple of π

Define a function: $f: \mathbb{R} \to \mathbb{Q}$ such that

(a) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

(b) $f(qx) = q\,f(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$

(c) $f(\pi) = 0$

Observe: $f\left(\frac{\pi}{2}\right) = \frac{1}{2} f(\pi) = 0$ and $f(\theta) = 0$ if θ is any rational multiple of π

For any edge e, let $\ell(e)$ be the length of e, and let $\phi(e)$ be the dihedral angle along e.

Then the **mass** of edge e is $\underbrace{\ell(e) \cdot f(\phi(e))}_{\text{note dependence of } f}$.

DEHN INVARIANT:

For any polyhedron P, let $D_f(P) = \sum_{e \in P} \ell(e) \cdot f(\phi(e))$

$\rightarrow$ **the Dehn invariant is invariant under dissections.**

Why? Let e be an edge that appears in a dissection of P. This can happen in 3 ways

(a) e is contained in an edge of P

The dihedral angles along e in the dissection add up the original dihedral angle along e.

The total mass along e is unchanged by dissection.

(b) e is contained in a face

Then the sum of the dihedral angles is π, and thus their masses add up to 0

(c) e is in the interior of P

Then the dihedral angles add up to 2π and the sum of the masses is 0.

If P and Q are polyhedra and $D_f(P) \neq D_f(Q)$, then P and Q are not scissors congruent.

$D_f(\text{cube}) = 0$

$D_f(\text{regular tetrahedron}) = 6 \cdot f\left(\arccos\left(\frac{1}{3}\right)\right) \neq 0$

if $f\left(\arccos\left(\frac{1}{3}\right)\right) \neq 0$ e.g. let $f\left(\arccos\left(\frac{1}{3}\right)\right) = 1$

Thus, the cube and the regular tetrahedron are <u>not</u> scissors congruent.