MOMENT-GENERATING FUNCTIONS (mgf)

mgf of rv X is $M_X(t) = E(e^{tX}) = \sum_x e^{tx} P(X=x)$

also: $M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{6} + \cdots$

moments of X

To find $E(X^r)$, differentiate $M_X(t)$ r times and set $t=0$.

EXAMPLE: Let $X \sim \text{Poisson}(\mu)$. Find the mgf of X.

Then: $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \underbrace{e^{-\mu}\frac{\mu^k}{k!}}_{P(X=k)}$

$= e^{-\mu} \sum_{k=0}^{\infty} e^{tk} \frac{\mu^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!}$

Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

let $x = \mu e^t$

$$\boxed{M_X(t) = e^{-\mu} e^{\mu e^t}}$$

Observe: $M_X(0) = e^{-\mu} e^{\mu} = 1$

$M'_X(t) = e^{-\mu} \cdot e^{\mu e^t} \cdot \mu e^t$, so $M'_X(0) = e^{-\mu} \cdot e^{\mu} \cdot \mu = \mu = E(X)$

the first moment

$M'_X(t) = \mu e^t e^{\mu(e^t - 1)}$

$M''_X(t) = \mu e^t e^{\mu(e^t-1)} + \mu e^t e^{\mu(e^t-1)} \mu e^t$

so $M''_X(0) = \mu e^0 e^{\mu(\cancel{e^0}^{1}-1)^{0}} + \mu e^0 e^{\mu(\cancel{e^0}^{1}-1)^{0}} \mu e^0$

$= \mu + \mu^2 = E(X^2)$

geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}$$

r = common ratio

$$\sum_{n=0}^{m-1} ar^n = a + ar + ar^2 + \cdots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$