

# Math 262

## Section 2.6.1

Day 14

1. Suppose that in a batch of 20 items, 3 are defective. If 5 of the items are sampled at random:

(a) What is the probability that none of the sampled items are defective?

(b) What is the probability that exactly 1 of the sampled items is defective?

(c) What is the probability that exactly 4 of the sampled items are defective?

(d) On average, how many defective items will be found in a random sample of 5 items?

(e) What is the probability that the number of defective items sampled is within 2 standard deviations of the mean number?

2. Let  $X$  be a hypergeometric random variable with parameters  $n$ ,  $M$ ,  $N$ . Let  $Y$  be a Binomial random variable with parameters  $n$  and  $p = \frac{M}{N}$ . How does  $E(X)$  compare to  $E(Y)$ ? How does  $\text{Var}(X)$  compare to  $\text{Var}(Y)$ ?

3. An unknown number,  $N$ , of animals inhabit a certain region. To estimate the size of the population, ecologists perform the following experiment: They first catch  $M$  of these animals, mark them in some way, and release them. After allowing the animals to disperse throughout the region, they catch  $n$  of the animals and count the number,  $X$ , of marked animals in this second catch.

The ecologists want to make a *maximum likelihood estimate* of the population size  $N$ . This means that if the observed value of  $X$  is  $x$ , then they estimate the population size to be the integer  $N$  that maximizes the probability that  $X = x$ . Help them complete this estimate as follows.

- (a) What assumptions are necessary to say that  $X$  has a hypergeometric distribution?
- (b) Let  $P_x(N)$  be the probability that  $X = x$  given that  $X \sim \text{Hypergeometric}(n, M, N)$ . Write down a formula for  $P_x(N)$ .
- (c) Simplify the ratio  $\frac{P_x(N)}{P_x(N-1)}$ . *Hint: use FullSimplify in Mathematica!*
- (d) Show that  $\frac{P_x(N)}{P_x(N-1)} \geq 1$  if and only if  $N \leq \frac{Mn}{x}$ .
- (e) Explain why  $P_x(N)$  attains its maximum value when  $N$  is the largest integer less than or equal to  $\frac{Mn}{x}$ . What is the most likely population size  $N$ ?
- (f) If  $M = 30$ ,  $n = 20$ , and  $x = 7$ , what is the maximum likelihood estimate for  $N$ ?

4. Urn 1 contains 100 balls, 10 of which are red. Let  $X_1$  be the number of red balls in a random sample of size 50 from Urn 1. Urn 2 contains 100 balls, 50 of which are red. Let  $X_2$  be the number of red balls in a random sample of size 10 from Urn 2.

- (a) Use technology to compute the pmf of  $X_1$ . Display the values as a list or a table. Then do the same for the pmf of  $X_2$ . What do you notice?
- (b) Change the numbers 100, 10, and 50 in this problem and recompute the pmfs of  $X_1$  and  $X_2$ . What do you notice?
- (c) Make a conjecture about when two hypergeometric random variables have the same pmf.

**BONUS:** Prove your conjecture.