Math 262

Section 2.3 Day 10

1. Suppose you have an unfair coin that lands on heads only 40% of the time. You flip this coin three times. Let X be the number of heads that appear in the three flips of the coin. The probability distribution of X is given by the pmf

$$p(x) = \binom{3}{x}(0.4)^x(0.6)^{3-x} \qquad \text{for} \qquad x \in \{0,1,2,3\}.$$

   (a) What is the expected value $E(X)$?

   (b) What is $E(X^2)$?

   (c) What is $\text{Var}(X)$? *Hint: use the shortcut formula!*

   (d) Suppose the coin flips are part of a game in which you win $5X+2$ dollars. Let $Y = 5X+2$. Show that the pmf of Y is given by:

y	2	7	12	17
$P(Y=y)$	0.216	0.432	0.288	0.064

   (e) Use the pmf of Y to find $E(Y)$, your expected winnings in this game.

   (f) Use the pmf of Y to find $E(Y^2)$, and then find is $\text{Var}(Y)$.

   (g) How is $E(Y)$ related to $E(X)$? How is $\text{Var}(Y)$ related to $\text{Var}(X)$?

(a) $E(X) = 0\cdot p(0) + 1\cdot p(1) + 2\cdot p(2) + 3\cdot p(3) = 0 + 1\cdot 3(0.4)(0.6)^2 + 2\cdot 3(0.4)^2(0.6) + 3\cdot 1(0.4)^3 = 1.2$

(b) $E(X^2) = 0^2\cdot p(0) + 1^2\cdot p(1) + 2^2\cdot p(2) + 3^2\cdot p(3) = 0 + 3(0.4)(0.6)^2 + 12(0.4)^2(0.6) + 9(0.4)^3 = 2.16$

(c) $\text{Var}(X) = E(X^2) - (E(X))^2 = 2.16 - 1.2^2 = 0.72$

(d) The values of Y are given by $5X-2$, for $X = 0,1,2,3$. The probability values are the same as those for X; that is, $P(Y=2) = P(X=0)$, $P(Y=7) = P(X=1)$, etc.

(e) $E(Y) = 2(0.216) + 7(0.432) + 12(0.288) + 17(0.064) = 8$

(f) $E(Y) = 2^2(0.216) + 7^2(0.432) + 12^2(0.288) + 17^2(0.064) = 82$, so $\text{Var}(Y) = E(Y^2) - (E(Y))^2 = 82 - 64 = 18$

(g) $E(Y) = E(5X+2) = 5E(X)+2$ and $\text{Var}(Y) = \text{Var}(5X+2) = 5^2\text{Var}(X)$
Expected value is a linear operator, but variance is not!

The following two problems require **Chebyshev's Inequality**: Let X be a discrete random variable with mean μ and standard deviation σ. For any $k \geq 1$,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

In words, *the probability that X is at least k standard deviations away from its mean is at most* $\frac{1}{k^2}$.

2. Verify that Chebyshev's Inequality holds for the random variable X from Problem 1, using the values $k=2$. That is, check that $P(|X-\mu| \geq 2\sigma)$ is less than $\frac{1}{(2)^2}$.

Note that $\sigma = \sqrt{\text{Var}(X)} = \sqrt{0.72} \approx 0.85$. Then:

$$P(|X-\mu| \geq 2\sigma) = P(|X-1.2| \geq 1.7) = P(X \leq -0.5) + P(X \geq 2.9) = 0 + 0.064 \leq 0.25 = \frac{1}{(2)^2}$$

Chebyshev's inequality says that the probability that X is *more* than 2 standard deviations from its mean is at most 0.25. This is true. Since we know the distribution of X, we confirmed that this probability is only 0.064.

3. The number of equipment breakdowns in a manufacturing plant averages 4 per week, with a standard deviation of 0.7 per week.

(a) Find an interval that includes at least 90% of the weekly figures for the number of breakdowns.

Hint: Apply Chebyshev's Inequality with k that solves $\frac{1}{k^2} = 0.1$.

If $\frac{1}{k^2} = 0.1$, then $k = \sqrt{10} = 3.16$. Thus, Chebyshev's Inequality gives:

$$P\left(|X - 4| \geq 3.16(0.7)\right) = P\left(|X - 4| \geq 2.21\right) = P(X \leq 1.79 \text{ or } X \geq 6.21) \leq 0.1$$

That is, the probability that X is *not* between 1.79 and 6.21 is not greater than 0.1. We want the probability of the complementary event: $P(1.79 < X < 6.21) > 0.9$. Thus, the interval $(1.79, 6.21)$ includes at least 90% of the numbers of weekly breakdowns.

(b) A plant supervisor promises that the number of breakdowns will rarely exceed 7 in a one-week period. Is the supervisor justified in making this claim? Why?

From the interval found above, we see that at least 90% of weeks will have less than 7 breakdowns.

We can do even better. Applying Chebyshev's Inequality with $k = 5$:

$$P\left(|X - 4| \geq 5(0.7)\right) = P(X \leq 0.5 \text{ or } X \geq 7.5) = P(X > 7) \leq \frac{1}{5^2} = 0.04$$

Thus, the probability that there are more than 7 breakdowns in a week is not greater than 0.4, so the supervisor is justified in making this claim.

4. When flipped, a certain coin comes up heads with probability p. Let X be the number of heads that appear in n flips of this coin.

(a) What is the probability distribution of X?

X takes on the values $0, 1, \ldots, n$. For any $k \in \{0, 1, \ldots, n\}$, there are $\binom{n}{k}$ sequences of heads and tails that include exactly k heads, and each such sequence occurs with probability $p^k(1-p)^{1-k}$. Thus:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{1-k}$$

and this specifies the probability distribution.

(b) Show that $E(X) = np$.

Hint: Write $E(X)$ as a sum and factor out np. Then use the binomial theorem to show that the sum equals 1.

$$\begin{aligned}
E(X) &= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} && \text{note that we can start at } k = 1 \\
&= np \sum_{k=1}^{n} k \frac{(n-1)!}{k!(n-k)!} p^{k-1} (1-p)^{n-k} && \text{factor out } np \text{ from the sum} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} && \text{simplify the } k\text{s} \\
&= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-j-1)!} p^{j} (1-p)^{n-j-1} && \text{let } j = k - 1 \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1-p)^{n-1-j} && \text{recognize the binomial coefficient} \\
&= np \left(p + (1-p)\right)^{n-1} && \text{apply the binomial theorem} \\
&= np(1)^{n-1} && \text{simplify} \\
&= np && \text{done!}
\end{aligned}$$