Math 262

Sections 2.1 and 2.2 Day 9

1. State the possible values of each random variable below, and say whether it is discrete or continuous. You might need to make some assumptions; if so, state your assumptions.

   (a) $X =$ the sum of the numbers that appear when two standard dice are rolled

   (b) $N =$ the number of defective circuit boards in a shipment

   (c) $T =$ the temperature in Northfield on a day in February

   (d) $Y =$ the annual income of a randomly-selected person in Minnesota

   (e) $L =$ the length of a fish caught in Lake Itasca

   (a) X is a discrete rv taking values in the set $\{2, 3, \ldots, 12\}$.

   (b) N is a discrete rv taking values in the set $\{0, 1, 2, \ldots\}$. If we assume that there is some maximum number of circuit boards in a shipment, then X has a maximum value.

   (c) T takes values in some interval, such as $(-40, 80)$ degrees Fahrenheit; thus T is a continuous rv.

   (d) Assuming that we measure income in dollars, Y is a discrete rv taking values in the nonnegative integers.

   (e) L is a continuous rv taking values in some interval, such as $(0, 2)$ meters.

2. Suppose that one out of every eight calls you receive is from a telemarketer. (Assume that all calls are independent.)

   (a) Let $X = 1$ if the next call you receive is from a telemarketer, and let $X = 0$ otherwise. What type of random variable is X? State the probability mass function (pmf) and cumulative distribution function (cdf) of X. Then sketch each function.

   (b) Let Y be the number of telemarketers in the next four phone calls. State the pmf and cdf of Y, and sketch each function.

   (c) Let Z be the number of calls you receive until (and including) the first call from a telemarketer. State the pmf and cdf of Z, and sketch each function.

   (a) Since the only possible values of X are 0 and 1, X is a Bernoulli random variable. The pmf of X is given by $p(0) = \frac{7}{8}$, $p(1) = \frac{1}{8}$, and $p(x) = 0$ otherwise. The cdf of X is given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.875 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

   (b) The pmf of Y is given by $p(y) = \binom{4}{y}\left(\frac{1}{8}\right)^y\left(\frac{7}{8}\right)^{4-y}$ for $y \in \{0, 1, \ldots, 4\}$. Decimal approximations of the nonzero values of its pmf are: $p(0) = 0.5862$, $p(1) = 0.3350$, $p(2) = 0.0718$, $p(3) = 0.0068$, $p(4) = 0.0002$. The cdf is approximately:

x	$(-\infty, 0)$	$[0, 1),$	$[1, 2)$	$[2, 3)$	$[3, 4)$	$[4, \infty)$
$F(y)$	0	0.5862	0.9212	0.9930	0.9998	1

   (c) The pmf of Z is given by $p(z) = \left(\frac{7}{8}\right)^{z-1}\left(\frac{1}{8}\right)$ for positive integer values of z. The cdf of Z is given by $F(z) = 1 - \left(\frac{7}{8}\right)^{\lfloor z \rfloor}$ for positive z, and $F(z) = 0$ for $z \le 0$.

3. The cdf for a random variable X is as follows:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \le x < 1 \\ 0.5 & 1 \le x < 2 \\ 0.8 & 2 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

(a) What is $P(X = 2)$?
(b) What is $P(X = 3)$?
(c) What is $P(2.5 \le X)$?
(d) Sketch the pmf of X.

(a) $P(X = 2) = F(2) - F(2-) = 0.8 - 0.5 = 0.3$
(b) $P(X = 3) = F(3) - F(3-) = 0.8 - 0.8 = 0$
(c) $P(2.5 \le X) = 1 - F(2.5-) = 1 - 0.8 = 0.2$
(d) The nonzero values of the pmf are $p(0) = 0.2$, $p(1) = 0.3$, $p(2) = 0.3$, and $p(4) = 0.2$.

4. Which of the following functions is the pmf for some random variable X?

(a) $p(x) = \frac{x^2}{50}$ for $x = 1, 2, \ldots, 5$
(b) $p(x) = \log_{10}\left(\frac{x+1}{x}\right)$ for $x = 1, 2, \ldots, 9$

(a) The function in (a) is not a pmf because $\sum_{x=1}^{5} \frac{x^2}{50} = \frac{55}{50} \neq 1$.
(b) The function in (b) is a pmf. The sum of its nonzero values is 1:

$$\sum_{x=1}^{9} \log_{10}\left(\frac{x+1}{x}\right) = log_{10}\left(\frac{2}{1} \cdot \frac{3}{2} \cdots \frac{10}{9}\right) = \log_{10}(10) = 1$$

(This is the pmf for the distribution known as Benford's Law.)

5. Which of the following properties must hold for any cdf $F(x)$? For each property, either say why it must hold or give a counterexample to show that it might not hold.

(a) $\lim_{b\to-\infty} F(b) = 0$
(b) $\lim_{b\to\infty} F(b) = 1$
(c) $F(x)$ is continuous
(d) $F(x)$ is nondecreasing; that is, if $a < b$, then $F(a) \le F(b)$.
(e) $F(b) = 0.5$ for some value b.

(a) Yes, because $P(X \le b) \to 0$ as $b \to -\infty$.
(b) Yes, because $P(X \le b) \to 1$ as $b \to \infty$.
(c) No — in the problems above, we have seen many cdfs that are not continuous functions. The cdf for a continuous rv is continuous, and we'll study those in Chapter 3.
(d) Yes. If $a < b$, then $F(a) = P(X \le a) \le P(X \le b) = F(b)$.
(e) No — consider the cdf in #2(a) above.

6. Three balls are randomly selected (without replacement) from an urn containing 20 balls numbered 1 through 20. Let X denote the largest number selected. What is $P(X \ge 17)$?

Suppose that each of the $\binom{20}{3}$ selections are equally likely.

The event $(X = i)$ occurs exactly when the ball numbered i is chosen along with two balls numbered 1 through $i - 1$. This can occur in $\binom{1}{1}\binom{i-1}{2}$ ways. Thus:

$$P(X \ge 17) = P(X = 17) + P(X = 18) + P(X = 19) + P(X = 20) = \frac{\binom{16}{2}}{\binom{20}{3}} + \frac{\binom{17}{2}}{\binom{20}{3}} + \frac{\binom{18}{2}}{\binom{20}{3}} + \frac{\binom{10}{2}}{\binom{20}{3}} = \frac{29}{57} \approx 0.508$$