Practice Problems on Transformations of Random Variables

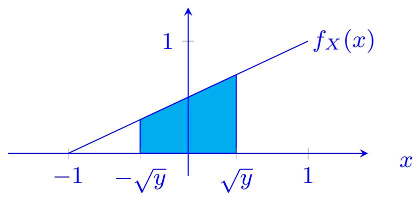
Math 262

1. Let X have pdf given by $f_X(x) = \frac{x+1}{2}$ for $-1 \le x \le 1$. Find the density of $Y = X^2$.

   Note that $0 \le Y \le 1$. Fix $y \in [0,1]$. Then:

   $$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

   This probability is equal to the shaded area below:

   The shaded region is a trapezoid with area $\sqrt{y}$, so $F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \sqrt{y}$. Differentiating, we find $f_Y(y) = \dfrac{d}{dy} F_Y(y) = \dfrac{1}{2\sqrt{y}}$ for $0 \le y \le 1$.

2. Let Y have pdf given by $f_Y(y) = 2(1-y)$ for $0 \le y \le 1$.

   (a) Find the density of $U_1 = 2Y - 1$.

   $U_1 = g_1(Y)$, where $g_1(y) = 2y - 1$. Since g_1 is monotonic, we can apply the Transformation Theorem. The inverse of g_1 is $h_1(u_1) = \frac{u_1+1}{2}$ for $-1 \le u_1 \le 1$. The density of U_1 is then:

   $$f_{U_1}(u_1) = f_Y\left(h_1(u_1)\right)\left|h_1'(u_1)\right| = 2\left(1 - \frac{u_1+1}{2}\right)\left|\frac{1}{2}\right| = \frac{1-u_1}{2} \qquad \text{for } -1 \le u_1 \le 1.$$

   (b) Find the density of $U_2 = 1 - 2Y$.

   $U_2 = g_2(Y)$, where $g_2(y) = 1 - 2y$. Since g_2 is monotonic, we can apply the Transformation Theorem. The inverse of g_2 is $h_2(u_2) = \frac{1-u_2}{2}$, for $-1 \le u_2 \le 1$. The density of U_2 is then:

   $$f_{U_2}(u_2) = f_Y\left(h_2(u_2)\right)\left|h_2'(u_2)\right| = 2\left(1 - \frac{1-u_2}{2}\right)\left|\frac{1}{2}\right| = \frac{1+u_2}{2} \qquad \text{for } -1 \le u_2 \le 1.$$

   (c) Find the density of $U_3 = Y^2$.

   $U_3 = g_3(Y)$, where $g_3(y) = y^2$, which *is* monotonic on the interval $0 \le y \le 1$, so we can apply the Transformation Theorem. The inverse of g_3 is $h_3(u_3) = \sqrt{u_3}$, for $0 \le u_3 \le 1$. The density of U_3 is then:

   $$f_{U_3}(u_3) = f_Y\left(h_3(u_3)\right)\left|h_3'(u_3)\right| = 2\left(1 - \sqrt{u_3}\right)\left|\frac{1}{2\sqrt{u_3}}\right| = \frac{1}{\sqrt{u_3}} - 1 \qquad \text{for } 0 \le u_3 \le 1.$$

3. Let $X \sim \text{Unif}[0,1]$. Find the density of $U = \sqrt{X}$.

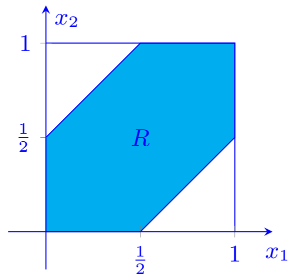
   First, $f_X(x) = 1$ for $0 \le x \le 1$. Since $U = g(X)$, where $g(x) = \sqrt{x}$, which is monotonic, we can apply the Transformation Theorem. The inverse of g is $h(u) = u^2$ for $0 \le u \le 1$. The density of U is then:

   $$f_U(u) = f_X\left(h(u)\right)\left|h'(u)\right| = 1\left|2u\right| = 2u \qquad \text{for } 0 \le u \le 1.$$

4. Two sentries are sent to patrol a road that is 1 mile long. The sentries are sent to points chosen independently and uniformly along the road. Find the probability that the sentries will be less than $\frac{1}{2}$ mile apart when they reach their assigned posts.

Let X_1 and X_2 be the posts of the sentries along the road; X_1 and X_2 are iid Unif$[0,1]$. Thus, their joint density is $f(x_1, x_2) = 1$ for $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$.

Let $Y = X_1 - X_2$. We want $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right)$. Note that we don't need the density of Y to answer the question: since the joint density of X_1 and X_2 is constant on the unit square, the probability $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right)$ is equal to the area of the shaded region R in the following figure.



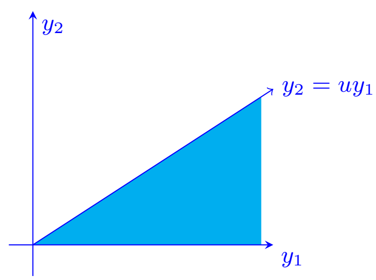
Thus, $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right) = \frac{3}{4}$.

5. The joint distribution for the lifetimes of two different types of components operating in a system is given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & \text{if } y_1 > 0, y_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the density function for the ratio $U = \frac{Y_2}{Y_1}$.

Using the (bivariate) distribution function method, first note that U can be any positive number. Fix $u > 0$, and note that the set of where $U = \frac{Y_2}{Y_1} = u$ in the $y_1 y_2$-plane is the line $y_2 = u y_1$.



The region where $U = \frac{Y_2}{Y_1} \le u$ is the region in the first quadrant where $y_2 \le u y_1$, which is the shaded region in the figure above.

Then, $P(U \le u) = P\left(\frac{Y_2}{Y_1} \le u\right) = \int_0^\infty \int_0^{uy_1} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 dy_1 = \frac{u^2+2u}{(1+u)^2}$.

Thus, the density of U is $f_U(u) = \frac{d}{du}\left(\frac{u^2+2u}{(1+u)^2}\right) = \frac{2}{(1+u)^3}$, for $u > 0$.

6. Suppose X and Y are independent exponential rvs with parameter λ. Find the joint density of $V = \frac{X}{Y}$ and $W = X + Y$. Use the joint density to find the marginal distributions.

We will use the bivariate tranformation theorem. Note that the joint density of X and Y is given by $f(x, y) = \lambda^2 e^{-\lambda(x+y)}$ for $x > 0$ and $y > 0$.

We must solve for X and Y in terms of V and W. Since $V = \frac{X}{Y}$, it follows that $X = VY$, and then $W = X + Y = VY + Y$, which we solve for Y to obtain $Y = \frac{W}{V+1}$. Similarly, we find $X = \frac{VW}{V+1}$. Thus, we have $X = \phi(V, W)$ where $\phi(v, w) = \frac{vw}{v+1}$, and $Y = \psi(V, W)$ where $\psi(v, w) = \frac{w}{v+1}$.

The Jacobian determinant is then:

$$|M| = \begin{vmatrix} \frac{\partial \phi}{\partial v} & \frac{\partial \phi}{\partial w} \\ \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{(v+1)^2} & \frac{v}{v+1} \\ \frac{-w}{(v+1)^2} & \frac{1}{v+1} \end{vmatrix} = \frac{w}{(v+1)^3} - \frac{-vw}{(v+1)^3} = \frac{w}{(v+1)^2}.$$

Therefore, the joint density of V and W is given by

$$g(v, w) = f(\phi(v, w), \psi(v, w))|M| = \lambda^2 e^{-\lambda\left(\frac{vw}{v+1} + \frac{w}{v+1}\right)} \left| \frac{w}{(v+1)^2} \right| = \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w}$$

for $v > 0$ and $w > 0$. Integrate to find the marginal densities:

$$g_V(v) = \int_0^\infty g(v, w)\, dw = \frac{1}{(1+v)^2}$$

and

$$g_W(w) = \int_0^\infty g(v, w)\, dv = \lambda^2 w e^{-\lambda w}.$$

7. Let X and Y have joint density $f(x, y)$. Let (R, Θ) be the polar coordinates of (X, Y).

(a) Give a general expression for the joint density of R and Θ.

Note that $R = \sqrt{X^2 + Y^2}$, $\Theta = \arctan\left(\frac{Y}{X}\right)$, $X = R\cos\Theta$, and $Y = R\sin\Theta$.

The Jacobian determinant is then:

$$|M| = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The joint density of R and Θ is given by:

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)|M| = f(r\cos\theta, r\sin\theta)r.$$

(b) Suppose X and Y are independent with $f(x) = 2x$ for $0 < x < 1$ and $f(y) = 2y$ for $0 < y < 1$. Use your result to find the probability that (X, Y) lies inside the circle of radius 1 centered at the origin.

The joint density of X and Y is given by $f(x, y) = 4xy$ for $0 < x < 1$ and $0 < y < 1$.

By the previous result, the joint density of R and Θ is given by

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)r = 4(r\cos\theta)(r\sin\theta)r = 4r^3\cos\theta\sin\theta.$$

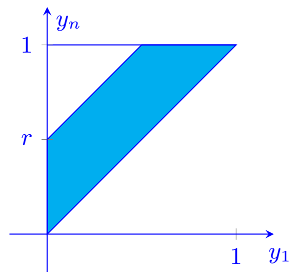
The point (X, Y) lies within the unit circle if and only if $R < 1$. Since both X and Y are positive, $0 < \Theta < \frac{\pi}{2}$, so the probability that $R < 1$ is given by

$$P(R < 1) = \int_0^{\pi/2} \int_0^1 4r^3 \cos\theta \sin\theta \, dr d\theta = \frac{1}{2}.$$

8. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the uniform distribution on $[0,1]$. Let Y_1 and Y_n be the smallest and largest, respectively, among the X_i. Find the pdf for the range $R = Y_n - Y_1$.

   Hint: The joint pdf for Y_1 and Y_n is $g(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}$ for $0 \le y_1 \le y_n \le 1$. (See exercise 141 in Chapter 4 of Carlton and Devore.)

   Since $0 \le R \le 1$, fix $r \in [0,1]$. Then $R = Y_n - Y_1 = r$ along the line $y_n = y_1 + r$ in the $y_1 y_n$-plane. Furthermore, $R \le r$ in the region below this line, which is the shaded region in the following diagram.



   Thus, the cdf of R is:

$$F_R(r) = P(R \le r) = 1 - \int_r^1 \int_0^{y_n - r} n(n-1)(y_n - y_1)^{n-2}\, dy_1 dy_n = (1-n)r^n + nr^{n-1}.$$

   Differentiate to find the pdf of R:

$$f_R(r) = \frac{d}{dr} F_R(r) = \frac{d}{dr}\left[(1-n)r^n + nr^{n-1}\right] = n(n-1)\left(r^{n-2} - r^{n-1}\right) \qquad \text{for } 0 \le r \le 1.$$