1. Let $\phi(x) = \alpha f(x) + \beta g(x)$. Under what conditions on the constants α and β will the $\phi(x)$ be a pdf for all possible pdfs $f(x)$ and $g(x)$?

Since f and g are pdfs:

$$f(x) \geq 0, \quad g(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} g(x)\,dx = 1$$

If $\alpha f(x) + \beta g(x)$ is a pdf, for all possible pdfs $f(x)$ and $g(x)$, it must be that $\alpha \geq 0$, $\beta \geq 0$, and:

$$1 = \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x)\,dx = \alpha \int_{-\infty}^{\infty} f(x)\,dx + \beta \int_{-\infty}^{\infty} g(x)\,dx = \alpha + \beta$$

So $\alpha + \beta = 1$.

2. Let $X \sim \text{Exp}(\lambda)$, $0 \leq s$ and $0 \leq t$. Since X is memoryless, is it true that $(X > s + t)$ and $(X > t)$ are independent events?

Memoryless Property: $P(X > s+t \mid X > t) = P(X > s)$

Since $P(X > s) \neq P(X > s+t)$, we have

$$P(X > s+t \mid X > t) \neq P(X > s+t),$$

so the events $X > s+t$ and $X > t$ are not independent.

3. Let $X_1, X_2, \dots, X_{10}$ be iid random variables denoting bids on an item that is for sale in an auction. The item will be sold to the highest bidder. If the bids are independent and uniformly distributed between 10 and 30, what is the expected value of the sale price?

Each X_i has pdf $f_X(x) = \frac{1}{20}$ and cdf $F_X(x) = \frac{x-10}{20}$ for $10 \leq x \leq 30$.

$Y_{10} = \max(X_i)$ has pdf $g_{10}(y) = 10\left[F_X(y)\right]^9 f_X(y) = 10\left[\frac{y-10}{20}\right]^9 \cdot \frac{1}{20} = \frac{(y-10)^9}{2(20)^9}$ for $10 \leq y \leq 30$.

Thus, $E(Y_{10}) = \int_{10}^{30} y \cdot \frac{(y-10)^9}{2(20)^9}\,dy = \frac{310}{11}$.

4. Suppose B and C are iid Unif[0,1]. Find the probability that the roots of the equation $x^2 + Bx + C = 0$ are real.

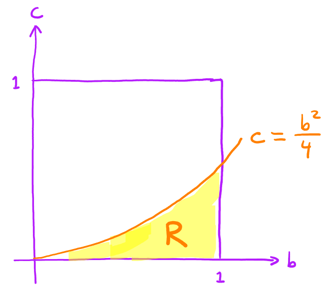
The roots are $x = \dfrac{-B \pm \sqrt{B^2 - 4C}}{2}$, which are real iff $B^2 - 4C \geq 0$,

or equivalently, $C \leq \frac{B^2}{4}$.

Then: $P(\text{real roots}) = P\left(C \leq \frac{B^2}{4}\right)$

$= \iint_R 1 \, dA = \text{Area}(R)$

$= \int_0^1 \frac{b^2}{4} \, db = \frac{1}{12}$



5. Alina makes 100 flips of a fair coin, and Dennis makes 99 flips of a fair coin. What is the probability that Alina gets *more* heads than Dennis?

Let A be the number of heads that Alina gets in the first 99 flips.

Let D be the number of heads that Dennis gets in 99 flips.

Let $q = P(A > D)$. By symmetry, $q = P(D > A)$ also.

Then $P(A = D) = 1 - 2q$.

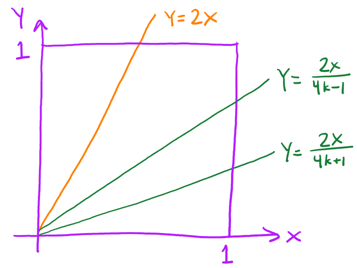
After 99 coin flips, Alina still gets one more coin flip.

Thus, Alina gets more heads than Dennis if $A > D$ or ($A = D$ and Alina's last flip is heads).

So: $P(\text{Alina gets more heads than Dennis}) = q + (1 - 2q) \cdot \frac{1}{2}$

$= q + \frac{1}{2} - q = \boxed{\frac{1}{2}}$

6. X and Y are iid Unif[0,1]. What is the probability that the closest integer to $\frac{X}{Y}$ is even?



(a) $\frac{X}{Y}$ is closest to zero if $\frac{X}{Y} < \frac{1}{2}$, or $2X < Y$.

$$P(2X < Y) = \frac{1}{4}$$

(b) If k is a positive integer, then $\frac{X}{Y}$ is closest to $2k$ if $\frac{4k-1}{2} < \frac{X}{Y} < \frac{4k+1}{2}$.

$$P\left(\frac{4k-1}{2} < \frac{X}{Y} < \frac{4k+1}{2}\right) = P\left(\frac{2X}{4k+1} < Y < \frac{2X}{4k-1}\right) = \frac{1}{4k-1} - \frac{1}{4k+1}$$

Combining parts (a) and (b), we have:

$$P(\text{integer closest to } \tfrac{X}{Y} \text{ is even}) = \frac{1}{4} + \sum_{k=1}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+1}\right)$$

← Mathematica can tell you this is $\frac{5-\pi}{4}$.

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \cdots$$

This sum is $1 - \frac{\pi}{4}$.

(Write the Taylor series for $\arctan(x)$ at $x = \frac{\pi}{4}$.)

$$= \frac{1}{4} + 1 - \frac{\pi}{4} = \boxed{\frac{5-\pi}{4}}$$