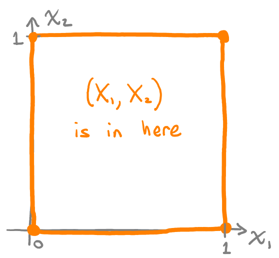
1. Suppose that X_1 and X_2 are iid Unif[0,1]. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) Find the region of possible values of the pair (Y_1, Y_2).

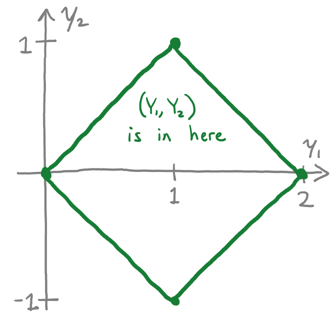
Note that the transformation is linear: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

Linear transformations map parallelograms to parallelograms.

The point (X_1, X_2) lies in a square. Let's see where the transformation maps this square:



$(0,0) \mapsto (0,0)$
$(1,0) \mapsto (1,1)$
$(1,1) \mapsto (2,0)$
$(0,1) \mapsto (1,-1)$



(b) Find the inverse transformation functions v_1 and v_2 such that $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Invert the transformation matrix: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

So $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, which means

$$X_1 = v_1(Y_1, Y_2) = \frac{Y_1 + Y_2}{2} \quad \text{and} \quad X_2 = v_2(Y_1, Y_2) = \frac{Y_1 - Y_2}{2}$$

(c) Use the transformation theorem to find the joint pdf of Y_1 and Y_2.

Jacobian matrix: $M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

So: $\det(M) = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

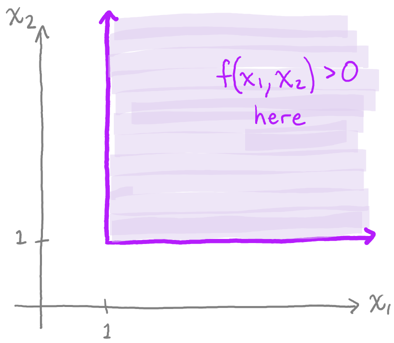
The joint density of Y_1 and Y_2 is:

$$g(y_1, y_2) = f(v_1(y_1, y_2), v_2(y_1, y_2)) \cdot |\det(M)| = 1 \cdot \left|\frac{1}{2}\right| = \frac{1}{2}$$

on the region found in part (a).

2. Let X_1 and X_2 have joint density $f(x_1, x_2) = \frac{1}{x_1^2 x_2^2}$ for $x_1 \geq 1$ and $x_2 \geq 1$. Let $Y_1 = X_1 X_2$ and $Y_2 = \frac{X_1}{X_2}$.

(a) Show that the region of positive joint density for Y_1 and Y_2 is given by $1 \leq Y_1$ and $\frac{1}{Y_1} \leq Y_2 \leq Y_1$.



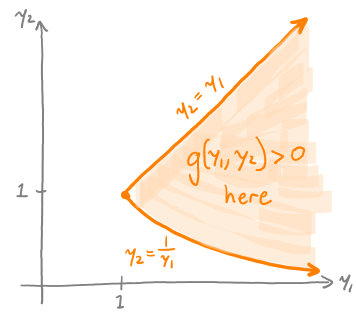
Note that Y_1 may be any value $y_1 \geq 1$.

For any fixed value $y_1 \geq 1$:

- the value $y_2 = \frac{x_1}{x_2}$ is smallest if $x_1 = 1$, meaning $y_2 = \frac{1}{y_1}$.
- the value $y_2 = \frac{x_1}{x_2}$ is largest if $x_2 = 1$, meaning $y_2 = y_1$.

Thus, the region of positive joint density for Y_1 and Y_2 is bounded by

$y_1 \geq 1$ and $\frac{1}{y_1} \leq y_2 \leq y_1$.



(b) Find the joint pdf of Y_1 and Y_2.

Find the inverse transformation:

$$Y_1 Y_2 = (X_1 X_2)\left(\frac{X_1}{X_2}\right) = X_1^2, \quad \text{so} \quad X_1 = \sqrt{Y_1 Y_2}. \quad \text{Thus} \quad v_1(y_1, y_2) = \sqrt{y_1 y_2}.$$

$$\frac{Y_1}{Y_2} = \frac{X_1 X_2}{\frac{X_1}{X_2}} = X_2^2, \quad \text{so} \quad X_2 = \sqrt{\frac{Y_1}{Y_2}} \quad \text{Thus} \quad v_2(y_1, y_2) = \sqrt{\frac{y_1}{y_2}}.$$

The Jacobian matrix:

$$M = \begin{bmatrix} \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} \\ \frac{1}{2\sqrt{y_1 y_2}} & \frac{-\sqrt{y_1}}{2y_2\sqrt{y_2}} \end{bmatrix} \quad \text{so} \quad \det(M) = \frac{\sqrt{y_2}}{2\sqrt{y_1}} \cdot \frac{-\sqrt{y_1}}{2y_2\sqrt{y_2}} - \frac{\sqrt{y_1}}{2\sqrt{y_2}} \cdot \frac{1}{2\sqrt{y_1 y_2}} = -\frac{1}{2y_2}$$

The Transformation Theorem says:

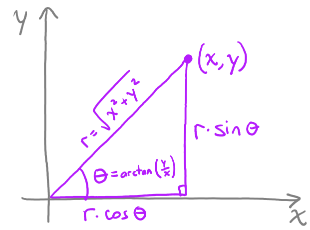
$$g(y_1, y_2) = f\left(\sqrt{y_1 y_2}, \sqrt{\frac{y_1}{y_2}}\right) \cdot \left|\frac{-1}{2y_2}\right| = \frac{1}{(y_1 y_2)\left(\frac{y_1}{y_2}\right)} \cdot \frac{1}{2y_2} = \frac{1}{2y_1^2 y_2} \quad \text{for } 1 \leq y_1, \ \frac{1}{y_1} \leq y_2 \leq y_1.$$

3. Let (X, Y) be a random point in the plane, where X and Y are independent standard normal random variables. Let (R, Θ) be the polar coordinates of (X, Y). Find the joint density of R and Θ. Then find the marginal densities of R and Θ. What is the probability that the point (X, Y) lies in a circle of radius 1 centered at the origin?

Transformations:

$$(X, Y) \xrightarrow{R = \sqrt{X^2+Y^2},\quad \Theta = \arctan\left(\frac{X}{Y}\right)} (R, \theta)$$

$$(R, \theta) \xrightarrow{X = R\cdot\cos\theta,\quad Y = R\cdot\sin\theta} (X, Y)$$



Region: X and Y may be any real numbers, so $R \geq 0$ and $0 \leq \Theta \leq 2\pi$.

Joint Density: $N(0,1)$ has pdf $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

X and Y are independent, so $f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$$f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

From the video for today, the transformation theorem gives:

$$g(r,\theta) = f(r\cdot\cos\theta,\ r\cdot\sin\theta)\cdot r$$

Thus: $$g(r,\theta) = \frac{1}{2\pi} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2}\cdot r$$

$$g(r,\theta) = \frac{r}{2\pi} e^{-r^2/2} \quad \text{for } r \geq 0,\ 0 \leq \theta < 2\pi$$

Marginal densities:

of θ: $$g_\theta(\theta) = \int_0^\infty \frac{r}{2\pi} e^{-r^2/2}\, dr = \frac{1}{2\pi} e^{-r^2/2} \Big|_{r=0}^{r=\infty} = \boxed{\frac{1}{2\pi} \quad \text{for } 0 \leq \theta \leq 2\pi}$$

$\theta \sim \text{Unif}[0, 2\pi]$

of R: $$g_r(\theta) = \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2}\, d\theta = \frac{r}{2\pi} e^{-r^2/2}\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{re^{-r^2/2} \quad \text{for } r \geq 0.}$$

R has a Rayleigh distribution

Inside circle:

$$P(R<1) = \int_0^1 re^{-r^2/2}\, dr = -e^{-r^2/2} \Big|_{r=0}^{r=1} = 1 - e^{-1/2} \approx 0.393$$