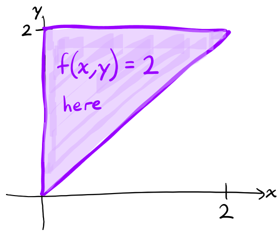
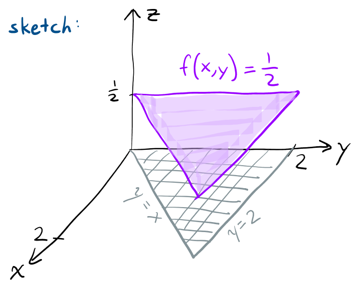
1. Let X and Y have joint density $f(x,y) = \frac{1}{2}$ for $0 \le x \le y \le 2$.

(a) Sketch the joint density of X and Y.

2D sketch:



3D sketch:



(b) What is the marginal density of X?

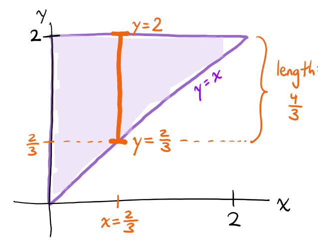
$$f_X(x) = \int_x^2 f(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy = \frac{y}{2}\Bigg|_{y=x}^{y=2} = \frac{2-x}{2} \quad \text{for } 0 \le x \le 2$$

(c) Suppose you know that $X = \frac{2}{3}$. What does $f\left(\frac{2}{3}, y\right)$ tell you about the density of Y, given that $X = \frac{2}{3}$?

If $x = \frac{2}{3}$, then $f(\frac{2}{3}, y) = \frac{1}{2}$ for $\frac{2}{3} \le y \le 2$.

Since $f(\frac{2}{3}, y)$ is constant, the density of $Y \mid X = \frac{2}{3}$ is constant for $\frac{2}{3} \le y \le 2$, which is an interval of length $\frac{4}{3}$.

Thus, $f_{Y|X}(y \mid \frac{2}{3}) = \frac{3}{4}$ for $\frac{2}{3} \le y \le 2$. That is, $Y \sim \text{Unif}[\frac{2}{3}, 2]$.



(d) Suppose you know that $X = x_0$. What is then the density of Y?

In this case, $Y \sim \text{Unif}[x_0, 2]$, so $f_{Y|X}(y \mid x_0) = \frac{1}{2-x_0}$ for $x_0 \le y \le 2$.

(e) In part (d), you found the conditional density $f_{Y|X}(\, y \mid x_0\,)$. How does this relate to the joint density $f(x,y)$ and the marginal density $f_X(x)$?

Observe that $f_{Y|X}(y \mid x_0) = \dfrac{1}{2-x_0} = \dfrac{\frac{1}{2}}{\frac{2-x_0}{2}} = \dfrac{f(x_0, y)}{f_X(x_0)}$

(f) If $X = \frac{2}{3}$, then what is the probability that $Y \le 1$?

$$P\left(Y \le 1 \mid X = \tfrac{2}{3}\right) = \int_{\frac{2}{3}}^{1} f_{Y|X}(y \mid \tfrac{2}{3})\,dy = \int_{\frac{2}{3}}^{1} \frac{3}{4}\,dy = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

Use the conditional density of Y given $X = \frac{2}{3}$.

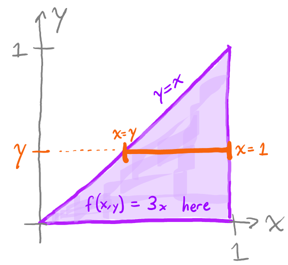
(g) What is the expected value of Y given that $X = x_0$?

mean of $\text{Unif}[x_0, 2]$

$$E(Y \mid X = x_0) = \int_{x_0}^{2} y \cdot f_{Y|X}(y \mid x_0)\,dy = \int_{x_0}^{2} y \cdot \frac{1}{2-x_0}\,dy = \frac{1}{2(2-x_0)}\, y^2 \Bigg|_{y=x_0}^{y=2} = \frac{4-x_0^2}{2(2-x_0)} = \frac{2+x_0}{2}$$

2. The joint pdf of X and Y is $f(x, y) = 3x$, for $0 \le y \le x \le 1$.

(a) What is the conditional distribution of X given $Y = y$?



$$f_Y(y) = \int_y^1 3x\,dx = \frac{3}{2}x^2\Big|_{x=y}^{x=1} = \frac{3}{2}(1-y^2) \quad \text{for} \quad 0 \le y \le 1$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3x}{\frac{3}{2}(1-y^2)} = \frac{2x}{1-y^2} \qquad 0 \le \underset{\text{fixed}}{y} \le \underset{\text{variable}}{x} \le 1$$

(b) What is $E(X \mid Y = y)$?

$$E(X|Y=y) = \int_y^1 x \cdot \frac{2x}{1-y^2}\,dx = \frac{2}{1-y^2}\int_y^1 x^2\,dx = \frac{2}{1-y^2}\cdot\frac{x^3}{3}\Big|_{x=y}^{x=1} = \frac{2}{1-y^2}\left(\frac{1}{3} - \frac{y^3}{3}\right) = \frac{2(1-y^3)}{3(1-y^2)}$$

(c) What is $\operatorname{Var}(X \mid Y = y)$?

$$E(X^2|Y=y) = \int_y^1 x^2 \cdot \frac{2x}{1-y^2}\,dx = \frac{1}{1-y^2}\cdot\frac{x^4}{2}\Big|_{x=y}^{x=1} = \frac{1-y^4}{2(1-y^2)} = \frac{1}{2}(1+y^2)$$

$$\operatorname{Var}(X^2|Y=y) = \frac{1}{2}(1+y^2) - \left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2 = \frac{(y-1)^2(y^2+4y+1)}{18(1+y)^2}$$

3. For continuous random variables X and Y, show that $E(E(X \mid Y)) = E(X)$.

inside: conditional expectation

outside: expectation of a function of Y

$$E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx, \quad \text{which is a function of } y$$

$$E(E(X|Y)) = \int_{-\infty}^{\infty} E(X|Y=y)\,f_Y(y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{\cancel{f_Y(y)}}\,dx\,\cancel{f_Y(y)}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x \underbrace{\boxed{\int_{-\infty}^{\infty} f(x,y)\,dy}}_{f_X(x)}\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = E(X)$$

4. The number of eggs N found in a nest of a certain species of turtle has a Poisson distribution with mean λ. Each egg has a probability p of being viable, and this event is independent from egg to egg. Find the mean and variance of the number of viable eggs per nest.

rvs: $N \sim \text{Poisson}(\lambda)$, X = number of viable eggs $\sim \text{Bin}(N, p)$

mean: $E(X) = E(\underbrace{E(X|N)}_{E(X|N) = Np}) = E(Np) = p\,E(N) = p\lambda$

variance: $\text{Var}(X) = \text{Var}(E(X|N)) + E(\text{Var}(X|N)) = \text{Var}(Np) + E(Np(1-p))$

$= p^2\,\text{Var}(N) + p(1-p)E(N) = p^2\lambda + p(1-p)\lambda = p^2\lambda - p^2\lambda + p\lambda = p\lambda$

BONUS: If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.

First, compute the conditional pmf of X given that $X+Y=m$.

$$P(X=k \mid X+Y=m) = \frac{P(X=k \text{ and } X+Y=m)}{P(X+Y=m)} = \frac{P(X=k)\,P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \cdot \binom{n}{m-k} p^{m-k} (1-p)^{n-m+k}}{\binom{2n}{m} p^m (1-p)^{2n-m}}$$

For the denominator, note that $X+Y \sim \text{Bin}(2n, p)$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

This is a hypergeometric probability: the probability of k successes in a sample of size m from a population with n successes and n failures.

So the conditional distribution of X, given that $X+Y=m$, is hypergeometric, and its mean is $E(X \mid X+Y=m) = \frac{m}{2}$.