discrete

conditional pmf: $p_{X|Y}(x \mid y) = \dfrac{p(x,y)}{p_Y(y)}$

Note similarity to conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

continuous

conditional pdf: $f_{X|Y}(x \mid y) = \dfrac{f(x,y)}{f_Y(y)}$

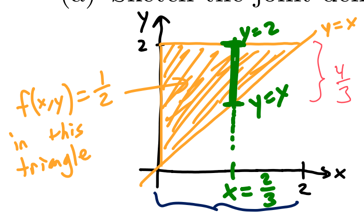
Use the conditional pmf or pdf to compute conditional expectation and conditional variance.

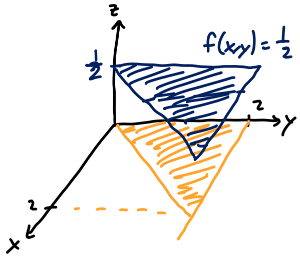
Math 262

Section 4.4 Day 32

1. Let X and Y have joint density $f(x,y) = \frac{1}{2}$ for $0 \le x \le y \le 2$.

   (a) Sketch the joint density of X and Y.





   (b) What is the marginal density of X?

$$f_X(x) = \int_x^2 f(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy = \frac{y}{2}\Big|_{y=x}^{y=2} = \frac{2}{2} - \frac{x}{2} = \boxed{\frac{2-x}{2} \text{ for } 0 \le x \le 2}$$

$f_X(x)$

   (c) Suppose you know that $X = \frac{2}{3}$. What does $f(\frac{2}{3}, y)$ tell you about the density of Y, given that $X = \frac{2}{3}$?

Since $f(\frac{2}{3}, y)$ is constant, the density of Y given $X = \frac{2}{3}$ is constant for $\frac{2}{3} \le y \le 2$.

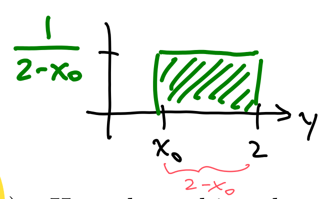
Specifically, $f_{Y|X}(y \mid \frac{2}{3}) = \frac{3}{4}$ for $\frac{2}{3} \le y \le 2$.

$X = \frac{2}{3}$

   (d) Suppose you know that $X = x_0$. What is then the density of Y?

If $X = x_0$, then $Y \sim \text{Unif}[x_0, 2]$.

So $f_{Y|X}(y \mid x_0) = \frac{1}{2-x_0}$ for $x_0 \le y \le 2$.



   (e) In part (d), you found the conditional density $f_{Y|X}(y \mid x_0)$. How does this relate to the joint density $f(x,y)$ and the marginal density $f_X(x)$?

Same

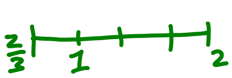
$$\frac{f(x_0, y)}{f_X(x_0)} = \frac{\frac{1}{2}}{\frac{2-x_0}{2}} \cdot \frac{2}{2} = \frac{1}{2-x_0} \text{ for } x_0 \le y \le 2$$

$= f_{Y|X}(y \mid x_0)$

   (f) If $X = \frac{2}{3}$, then what is the probability that $Y \le 1$?

$$P(Y \le 1 \mid X = \tfrac{2}{3}) = \int_{\frac{2}{3}}^{1} f_{Y|X}(y \mid \tfrac{2}{3})\,dy = \int_{\frac{2}{3}}^{1} \frac{3}{4}\,dy = \frac{3}{4} y \Big|_{\frac{2}{3}}^{1} = \frac{3}{4}\left(1 - \frac{2}{3}\right)$$

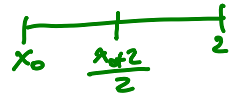
$$= \frac{3}{4} \cdot \frac{1}{3} = \boxed{\frac{1}{4}}$$



   (g) What is the expected value of Y given that $X = x_0$?

$$E(Y \mid X = x_0) = \int_{x_0}^{2} y \cdot f_{Y|X}(y \mid x_0)\,dy = \int_{x_0}^{2} y \cdot \frac{1}{2-x_0}\,dy = \frac{1}{2-x_0} \int_{x_0}^{2} y\,dy$$

$$= \frac{1}{2-x_0} \cdot \frac{1}{2} y^2 \Big|_{x_0}^{2} = \frac{1}{2-x_0} \cdot \frac{1}{2}\left(2^2 - x_0^2\right) = \frac{2+x_0}{2}$$



2. The joint pdf of X and Y is $f(x, y) = 3x$, for $0 \le y \le x \le 1$.

   (a) What is the conditional distribution of X given $Y = y$?

   (b) What is $E(X \mid Y = y)$?

   (c) What is $\text{Var}(X \mid Y = y)$?

3. For continuous random variables X and Y, show that $E(E(X \mid Y)) = E(X)$.

4. The number of eggs N found in nests of a certain species of turtles has a Poisson distribution with mean λ. Each egg has probability p of being viable, and this event is independent from egg to egg. Find the mean and the variance of the number of viable eggs per nest.

★ **BONUS:** If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.