Worksheet Solutions
Math 262 ⬩ 17 November 2023

1. Let $X \sim \text{Unif}[-1,1]$ and $Y = X^2$.

(a) Compute $E(X)$, $E(Y)$, and $E(XY)$. Does $E(XY) = E(X)E(Y)$?

$E(X) = 0$, $\quad E(Y) = E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\,dx = \frac{1}{6}x^3 \Big|_{-1}^{1} = \frac{1}{3}$

$E(XY) = E(X^3) = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\,dx = \frac{1}{8}x^4 \Big|_{-1}^{1} = 0$

Yes, $E(XY) = E(X)E(Y)$

(b) Are X and Y independent? Why or why not?

No: the value of X determines Y.

ANOTHER EXAMPLE: $U \sim \text{Unif}[0, 2\pi]$, $X = \cos(U)$, $Y = \sin(U)$

I. Two standard, fair dice are rolled. Let X_1 and X_2 be the numbers that appear on the dice.
II. An urn contains balls labeled 1, 2, 3, 4, 5, 6. Let Y_1 and Y_2 be the numbers on two balls drawn without replacement from the urn.

2. What is the distribution of X_i? How about the distribution of Y_i?

X_i and Y_i are both uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

3. What are $E(X_i)$ and $\text{Var}(X_i)$? How about $E(Y_i)$ and $\text{Var}(Y_i)$?

$E(X_i) = \frac{7}{2}$, $\quad E(X_i)^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

$\text{Var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$

Since X_i and Y_i have the same distribution,

$$E(Y_i) = \frac{7}{2} \quad \text{and} \quad Var(Y_i) = \frac{35}{12}$$

4. What are $E(X_1 + X_2)$ and $\text{Var}(X_1 + X_2)$?

By linearity, $E(X_1 + X_2) = E(X_1) + E(X_2) = 7$

Since X_1 and X_2 are independent,

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) = \frac{35}{6}$$

5. What and $E(Y_1 + Y_2)$ and $\text{Var}(Y_1 + Y_2)$?

By linearity, $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7$

Since Y_1 and Y_2 are dependent:

$$Var(Y_1 + Y_2) = Var(Y_1) + Var(Y_2) + 2\,Cov(Y_1, Y_2)$$

$$= \frac{35}{12} + \frac{35}{12} + 2\left(\frac{-7}{12}\right) = \frac{56}{12} = \frac{14}{3}$$

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{35}{3} - \frac{7}{2}\cdot\frac{7}{2} = -\frac{7}{12}$$

Possible products:

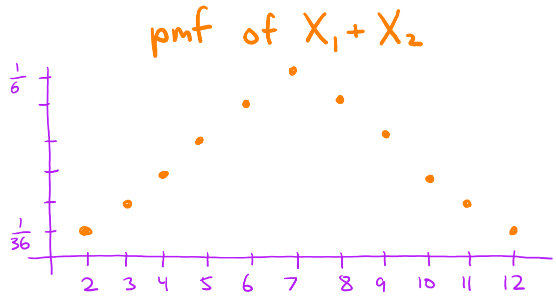
	1	2	3	4	5	6
1		2	3	4	5	6
2	2		6	8	10	12
3	3	6		12	15	18
4	4	8	12		20	24
5	5	10	15	20		30
6	6	12	18	24	30	

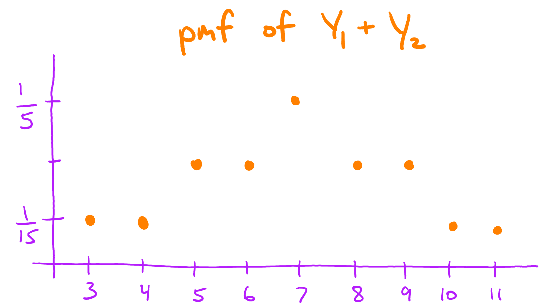
← All equally likely.

$$E(Y_1 Y_2) = \frac{1}{15}(2+3+4+5+6+6+8+10+12+12+15+18+20+24+30)$$

$$= \frac{175}{15} = \frac{35}{3}$$

6. Sketch the pmfs of $X_1 + X_2$ and $Y_1 + Y_2$. How does this help make sense of the means and variances that you found for these sums?





7. Generalize to rolls of n dice: find $E(X_1+\cdots+X_n)$ and $\mathrm{Var}(X_1+\cdots+X_n)$.

$$E(X_1+\cdots+X_n)=E(X_1)+\cdots+E(X_n)=\frac{7n}{2}$$

by independence, $\mathrm{Var}(X_1+\cdots+X_n)=\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n)=\frac{35n}{12}$

8. Similarly, generalize to choosing n balls from the urn. Find $E(Y_1+\cdots+Y_n)$ and $\mathrm{Var}(Y_1+\cdots+Y_n)$.

Now $n\le 6$.

As before, $E(Y_1+\cdots+Y_n)=\frac{7n}{2}$

However, now $\mathrm{Var}(Y_1+\cdots+Y_n)=\sum_{i=1}^{n}\mathrm{Var}(Y_i)+2\sum_{i<j}\mathrm{Cov}(Y_i,Y_j)$

$$=\frac{35n}{12}+2\,\frac{n^2-n}{2}\left(-\frac{7}{12}\right)=\frac{35n-7n^2+7n}{12}=\frac{42n-7n^2}{12}=\frac{7n(6-n)}{12}$$

Note that if $n=6$, the variance is zero.