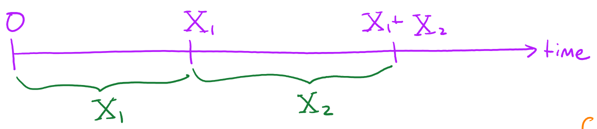
1. Suppose that the time between goals in a hockey game is exponentially distributed with mean 18 minutes (ignore timeouts and stoppages). Let X be the time from the start of the game until the second goal occurs.



X_1 and X_2 are each $Exp\left(\frac{1}{18}\right)$

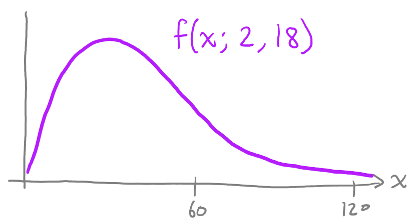
$\lambda = \frac{1}{18}$

[PREVIEW OF CH. 4: independent random variables]

$\alpha = 2, \ \beta = \frac{1}{\lambda} = 18$

Let $X = X_1 + X_2$. Then $X \sim Gamma(2, 18)$.

(a) Sketch the pdf of X.



(b) What is the probability that the second goal occurs less than 30 minutes after the game starts?

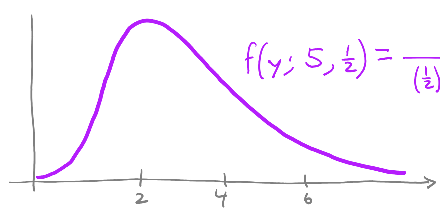
Then $P(X < 30) = \text{pgamma}\left(30, 2, \frac{1}{18}\right) = 0.496$

2. Suppose that a call center receives calls according to a Poisson distribution at a rate of 2 calls per minute. Let Y be the time between 10:00am and the 5^{th} call received after 10:00am.

Time between calls is $Exp(2)$.

$Y \sim Gamma\left(5, \frac{1}{2}\right)$ is the sum of five $Exp(2)$ rvs.

(a) Sketch the pdf of Y.



$f\left(y; 5, \frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^5 \Gamma(5)} y^4 e^{-2y} = \frac{32}{24} y^4 e^{-2y} = \frac{4}{3} y^4 e^{-2y}$

(b) What are the mean and variance of Y?

$E(Y) = 5 \cdot \frac{1}{2} = \frac{5}{2}$

$Var(Y) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}$

(c) What is $P(Y < 1)$?

$2 = \frac{1}{\beta}$

$\int_0^1 \frac{4}{3} y^4 e^{-2y} \, dy = \text{pgamma}(1, 5, 2) = 0.0526$

3. For large α, the gamma distribution converges to a normal distribution with mean $\alpha\beta$ and variance $\alpha\beta^2$. Investigate this in the case that $\beta = 1$.

(a) Let $X \sim \text{Gamma}(10, 1)$. Use technology to compute $P(X \leq x)$ for various values of x.

```
Table[CDF[GammaDistribution[10, 1], x], {x, 4, 16, 2}] // N
{0.00813224, 0.083924, 0.283376, 0.54207, 0.757608, 0.890601, 0.956702}
```

(b) Let $Z \sim N\left(10, \sqrt{10}\right)$. Use technology to compute $P(Z \leq x)$ for the same values of x that you used in part (a). Do you find the probabilities to be close to what you found in part (a)?

```
Table[CDF[NormalDistribution[10, Sqrt[10]], x], {x, 4, 16, 2}] // N
{0.0288898, 0.102952, 0.263545, 0.5, 0.736455, 0.897048, 0.97111}
```

These probabilities are somewhat close to those in part (a).

(c) Now choose a larger value of α, such as $\alpha = 100$. Compute some probabilities to verify that $X \sim \text{Gamma}(\alpha, 1)$ has nearly the same distribution as $Z \sim N\left(\alpha, \sqrt{\alpha}\right)$.

```
g = Table[CDF[GammaDistribution[100, 1], x], {x, 80, 140, 10}] // N
{0.0171083, 0.158221, 0.513299, 0.841721, 0.972136, 0.99725, 0.999839}

n = Table[CDF[NormalDistribution[100, 10], x], {x, 80, 140, 10}] // N
{0.0227501, 0.158655, 0.5, 0.841345, 0.97725, 0.99865, 0.999968}

g / n
{0.752009, 0.997263, 1.0266, 1.00045, 0.994767, 0.998598, 0.999871}
```

4. The skewness coefficient of the distribution of random variable X is defined

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}$$

How could you compute the skewness of $X \sim \text{Gamma}(\alpha, \beta)$? Then compute the skewness of X.

option 1: moments $f_X(t) = (1-\beta t)^{-\alpha}$

$$E(X) = \alpha\beta, \quad E(X^2) = \alpha(\alpha+1)\beta^2, \quad E(X^3) = \alpha(\alpha+1)(\alpha+2)\beta^3$$

From March 18: $$\gamma = \frac{E(X^3) - 3E(X^2)E(X) + 2E(X)^3}{\sigma^3}$$

$$\gamma = \frac{\alpha(\alpha+1)(\alpha+2)\beta^3 - 3(\alpha\beta(\alpha+1))(\alpha\beta) + 2\alpha^3\beta^3}{(\alpha\beta^2)^{3/2}} = \frac{2}{\sqrt{\alpha}}$$

NOTE: The skewness depends only on the shape parameter α.

option 2: integrate $\int_0^\infty (x-\alpha\beta)^3 \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx = 2\beta^3\alpha$

so $\gamma = \frac{2\beta^3\alpha}{(\alpha\beta^2)^{3/2}} = \frac{2}{\sqrt{\alpha}}$

option 3: Mathematica

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Skewness[GammaDistribution[α, β]]
```

$$\frac{2}{\sqrt{\alpha}}$$

BONUS: Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx = \int_0^\infty \frac{\sqrt{2}}{y} e^{-y^2/2}\, y\, dy = \sqrt{2}\int_0^\infty e^{-y^2/2}\, dy = \sqrt{2}\cdot\frac{\sqrt{\pi}}{\sqrt{2}} = \sqrt{\pi}$$

substitute: $y = \sqrt{2x}$

$x = \frac{y^2}{2}$

$dx = y\, dy$

standard normal pdf:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2/2}\, dx = 1$$

so $\int_0^\infty e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2} = \frac{\sqrt{\pi}}{\sqrt{2}}$