1. Let X represent the number of insurance policies sold by an agent in a day. The moment generating function of X is $M_X(t) = \underbrace{0.45}\underbrace{e^t} + \underbrace{0.35}\underbrace{e^{2t}} + \underbrace{0.15}\underbrace{e^{3t}} + \underbrace{0.05}\underbrace{e^{4t}}$, for $-\infty < t < \infty$. Calculate the standard deviation of X.

$\boxed{\sum e^{tx} P(X=x)}$ $e^{t\cdot 1}$ $x=1$ $x=2$ $x=3$ $x=4$

Differentiate: $M_X'(t) = 0.45e^t + 0.7e^{2t} + 0.45e^{3t} + 0.2e^{4t}$

Thus, $M_X'(0) = 0.45 + 0.7 + 0.45 + 0.2 = 1.8 = E(X)$

Differentiate again: $M_X''(t) = 0.45e^t + 1.4e^{2t} + 1.35e^{3t} + 0.8e^{4t}$

$M_X''(0) = 0.45 + 1.4 + 1.35 + 0.8 = 4 = E(X^2)$

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = 4 - 1.8^2 = 0.76$, so $\sigma_X = \sqrt{0.76} \approx \boxed{0.87}$

2. What do you think is the distribution of X in problem 1?

pmf: $p(1) = 0.45$, $p(2) = 0.35$, $p(3) = 0.15$, $p(4) = 0.05$

Note that $M_X(0)$ adds up the probabilities:

$M_X(0) = 0.45e^0 + 0.35e^{2(0)} + 0.15e^{3(0)} + 0.05e^{4(0)} = 1$

3. The skewness coefficient of the distribution of a random variable X is

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}$$

$(X-\mu)^3 = X^3 - 3X^2\mu + 3X\mu^2 - \mu^3$

The skewness is 0 if the distribution is symmetric, positive if the distribution is skewed right, or negative if the distribution is skewed left.

(a) Expand $(X-\mu)^3$ and use this to express $E[(X-\mu)^3]$ in terms of the moments $E(X)$, $E(X^2)$, and $E(X^3)$. Then express γ in terms of these moments.

First: $(X-\mu)^3 = X^3 - 3X^2\mu + 3X\mu^2 - \mu^3$

Then: $E[(X-\mu)^3] = E(X^3) - 3E(X^2)\mu + 3E(X)\mu^2 - \mu^3$

$= E(X^3) - 3E(X^2)E(X) + 3E(X)E(X)^2 - E(X)^3$

$= E(X^3) - 3E(X^2)E(X) + 2E(X)^3$

Also: $\sigma = \sqrt{E(X^2) - E(X)^2}$

Therefore: $$\gamma = \frac{E(X^3) - 3E(X^2)E(X) + 2E(X)^3}{\left(E(X^2) - E(X)^2\right)^{\frac{3}{2}}}$$

(b) For each of the following random variables, use Mathematica to compute the first three moments from the mgf. Then compute the skewness coefficient. Does the skewness coefficient agree with what you know about the shape of the distribution?

- $X \sim \text{Bin}\left(10, \frac{1}{2}\right)$

```
In[1]:= mx[t_] := (1/2 + 1/2 Exp[t])^10
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3)
Out[2]= 0
```

← This is the numerator of γ.

$\gamma = 0$, which means the distribution is symmetric

- $X \sim \text{Bin}\left(10, \frac{3}{4}\right)$

```
In[3]:= mx[t_] := (1/4 + 3/4 Exp[t])^10
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
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Out[4]= $-\sqrt{\frac{2}{15}}$

$\gamma = -\sqrt{\frac{2}{15}} < 0$, so the distribution is skewed left

- $X \sim \text{Geometric}\left(\frac{1}{3}\right)$

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In[5]:= mx[t_] := Exp[t] / (3 - 2 Exp[t])
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
```

Out[6]= $\frac{5}{\sqrt{6}}$

$\gamma = \frac{5}{\sqrt{6}} > 0$, so the distribution is skewed right

- $X \sim \text{Poisson}(4)$

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In[7]:= mx[t_] := Exp[4 (Exp[t] - 1)]
        (mx'''[0] - 3 mx''[0] * mx'[0] + 2 (mx'[0])^3) / (mx''[0] - mx'[0]^2)^(3/2)
```

Out[8]= $\frac{1}{2}$

$\gamma = \frac{1}{2} > 0$, so the distribution is skewed right

4. The monthly amount of time X (in hours) during which a manufacturing plant is inoperative due to equipment failures or power outages follows approximately a distribution with the following moment generating function:

$$M_X(t) = \left(\frac{1}{1-7.5t}\right)^2 = (1-7.5t)^{-2}$$

The amount of loss in profit due to the plant being inoperative is given by $Y = 12X + 1.25X^2$. Determine the variance of the loss in profit.

$M_X'(t) = 15(1-7.5t)^{-3}$ $\quad$ $M_X'(0) = 15 = E(X)$

$M_X''(t) = 337.5(1-7.5t)^{-4}$ $\quad$ $M_X''(0) = 337.5 = E(X^2)$

$M_X^{(3)}(t) = 10125(1-7.5t)^{-5}$ $\quad$ $M_X^{(3)}(0) = 10125 = E(X^3)$

$M_X^{(4)}(t) = 379687.5(1-7.5t)^{-6}$ $\quad$ $M_X^{(4)}(0) = 379687.5 = E(X^4)$

$E(Y) = 12E(X) + 1.25E(X^2) = 12(15) + 1.25(337.5) = 601.875$

$E(Y^2) = E\left(144X^2 + 30X^3 + \frac{25}{16}X^4\right) = 144(337.5) + 30(10125) + \frac{25}{16}(379687.5) = 945612$

$Var(Y) = E(Y^2) - E(Y)^2 = 945612 - (601.875)^2 = 583358$