1. Let $X \sim \text{Geometric}(p)$. Compute the mgf $M_X(t)$. Try to write the mgf without a summation.

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \underbrace{(1-p)^{x-1}p}_{P(X=x)} = \frac{p}{1-p} \sum_{x=1}^{\infty} e^{tx}(1-p)^x$$

$$\boxed{\text{Geometric series: } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

$$= \frac{p}{1-p} \underbrace{\sum_{x=1}^{\infty} \left(e^t(1-p)\right)^x}_{\text{geometric series: } a = r = e^t(1-p)}$$

$$= \frac{p}{\cancel{1-p}} \cdot \frac{e^t \cancel{(1-p)}}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)}$$

2. Suppose random variable X has probability mass function $P(X = x) = \frac{27}{40}\left(\frac{1}{3}\right)^x$, for integers $0 \le x \le 3$.

(a) Verify that this is a valid probability mass function.

$$\boxed{a + ar + ar^2 + \dots + ar^{n-1} = a\frac{1-r^n}{1-r}}$$

- nonnegative probabilities
- sum is: $\frac{27}{40}\underbrace{\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right)}_{\text{4 terms of a geometric series with } r=\frac{1}{3}} = \frac{27}{40}\left(\frac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right) = \frac{27}{40}\left(\frac{\frac{80}{81}}{\frac{2}{3}}\right) = \frac{27}{40}\cdot\frac{40}{27} = 1$

(b) Find the moment generating function $M_X(t)$. Try to write the mgf without a summation.

$$M_X(t) = E(e^{tX}) = e^0 \cdot \frac{27}{40} + e^t \cdot \frac{9}{40} + e^{2t} \cdot \frac{3}{40} + e^{3t} \cdot \frac{1}{40} = \frac{27}{40} \cdot \frac{1-\left(\frac{e^t}{3}\right)^4}{1-\left(\frac{e^t}{3}\right)} = \frac{81-e^{4t}}{40(3-e^t)}$$

common ratio: $r = \frac{e^t}{3}$

(c) Compute $M_X'(0)$. Does your answer agree with the expected value of X computed directly from the pmf?

$$M_X(t) = \frac{81-e^{4t}}{40(3-e^t)} \qquad M_X'(t) = \frac{40(3-e^t)(-4e^{4t}) - (81-e^{4t})(40)(-e^t)}{40^2(3-e^t)^2}$$

Use technology!

$$M_X'(0) = \frac{40(2)(-4) - (80)(40)(-1)}{40^2(2)^2} = \frac{72}{160} = \frac{9}{20}$$

$$E(X) = 0 + 1\cdot\frac{9}{40} + 2\cdot\frac{3}{40} + 3\cdot\frac{1}{40} = \frac{9+6+3}{40} = \frac{9}{20}$$

same!

This problem is in the next class video.

3. Let X be a discrete random variable with mgf $M_X(t)$, and let $Y = aX + b$. Write the definition of $M_Y(t)$. Replace Y with $aX + b$ and simplify the expected value to show that $M_Y(t) = e^{tb} M_X(at)$.

$$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{atX+bt}) = e^{bt} E(e^{atX}) = e^{bt} M_X(at)$$

$$\boxed{\text{If } Y = aX + b, \text{ then } M_Y(t) = e^{bt} M_X(at).}$$

We will do problems #4 and #5 next week.