

$$\rightarrow P(X=k) = (1-p)^{k-1} \cdot p$$

Last time: If  $X \sim \text{Geometric}(p)$  and  $k \in \mathbb{Z}^+$ ,  
then  $P(X > k) = (1-p)^k$ .

← geometric tail probability

Now: If  $X > 5$ , what is the probability that  $X > 8$ ?

$$P(X > 8 \mid X > 5) = \frac{P(X > 8 \text{ and } X > 5)}{P(X > 5)} = \frac{P(X > 8)}{P(X > 5)}$$

$$= \frac{(1-p)^8}{(1-p)^5} = (1-p)^3 = P(X > 3)$$

More generally, for integers  $0 < s < t$ :

$$P(X > t \mid X > s) = P(X > t-s)$$

MEMORYLESS  
PROPERTY

of a geometric  
random variable

The waiting time until the next success does  
not depend on how many failures you have already seen.

## MOMENT-GENERATING FUNCTIONS (mgf)

mgf of random variable  $X$

$$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{3!} + \dots$$

$$= E(e^{tX}) = \sum_x e^{tx} P(X=x)$$

To find  $E(X^r)$ , differentiate  $M_X(t)$   $r$  times and  
evaluate at  $t=0$ .

EXAMPLE:  $X \sim \text{Poisson}(\mu)$

$$P(X=k) = e^{-\mu} \frac{\mu^k}{k!}$$

Find mgf:  $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} e^{-\mu} \frac{\mu^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!}$

$= e^{-\mu} e^{\mu e^t} = e^{-\mu + \mu e^t}$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$   $x = \mu e^t$

$$M_X(t) = e^{\mu(e^t - 1)}$$

If we differentiate  $M_X(t)$ , do we recover  $E(X^r)$ ?

always true:  
 $M_X(0) = 1$

$$M_X(0) = e^{\mu(e^0 - 1)} = e^{\mu(0)} = e^0 = 1 = E(X^0)$$

$$M'_X(t) = e^{\mu(e^t - 1)} \cdot \mu e^t$$

$$\text{so } M'_X(0) = e^{\mu(e^0 - 1)} \cdot \mu e^0 = e^0 \cdot \mu e^0 = \mu = E(X)$$

$$M''(t) = e^{t + \mu(e^t - 1)} \cdot \mu(1 + \mu e^t)$$

$$\text{so } M''(0) = e^{0 + \mu(1 - 1)} \cdot \mu(1 + \mu e^0) = e^0 \cdot \mu(1 + \mu) = \mu + \mu^2 = E(X^2)$$

①  $X \sim \text{Geo}(p)$   $P(X=k) = (1-p)^{k-1} p$

$$M_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p$$

$$= e^t (1-p)^0 p + e^{2t} (1-p)^1 p + e^{3t} (1-p)^2 p + \dots$$

$k=1 \qquad k=2 \qquad k=3$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

geometric series  
common ratio:  $e^t(1-p) = r$   
initial term:  $e^t(1-p)^0 p = pe^t = a$

$$\rightarrow \frac{a}{1-r} \rightarrow \frac{pe^t}{1 - e^t(1-p)} = M_X(t)$$