$P(X=k) = (1-p)^{k-1} \cdot p$

Last time: If $X \sim \text{Geometric}(p)$ and $k \in \mathbb{Z}^+$, then $P(X>k) = (1-p)^k$.

geometric tail probability

Now: If $X>5$, what is the probability that $X>8$?

$$P(X>8 \mid X>5) = \frac{P(X>8 \text{ and } X>5)}{P(X>5)} = \frac{P(X>8)}{P(X>5)}$$

$$= \frac{(1-p)^8}{(1-p)^5} = (1-p)^3 = P(X>3)$$

More generally, for integers $0<s<t$:

$$\boxed{P(X>t \mid X>s) = P(X>t-s)}$$

MEMORYLESS PROPERTY of a geometric random variable

The waiting time until the next success does not depend on how many failures you have already seen.

MOMENT-GENERATING FUNCTIONS (mgf)

mgf of random variable X

moments of X

$$M_X(t) = 1 + E(X)\,t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{3!} + \cdots$$

$$= E(e^{tX}) = \sum_x e^{tx} P(X=x)$$

To find $E(X^r)$, differentiate $M_X(t)$ r times and evaluate at $t=0$.

EXAMPLE: $X \sim \text{Poisson}(\mu)$ $\qquad$ $P(X=k) = e^{-\mu}\frac{\mu^k}{k!}$

Find mgf: $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}\, \boxed{e^{-\mu}\frac{\mu^k}{k!}} = e^{-\mu}\sum_{k=0}^{\infty}\frac{(\mu e^t)^k}{k!}$

$x = \mu e^t$

$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$= e^{-\mu}e^{\mu e^t} = e^{-\mu+\mu e^t}$$

$$M_X(t) = e^{\mu(e^t-1)}$$

If we differentiate $M_X(t)$, do we recover $E(X^r)$?

always true: $M_X(0) = 1$

$$M_X(0) = e^{\mu(e^0-1)} = e^{\mu(0)} = e^0 = 1 = E(\overset{1}{X^0})$$

$$M_X'(t) = e^{\mu(e^t-1)}\cdot \mu e^t$$

so $M_X'(0) = e^{\mu(e^0-1)}\cdot \mu e^0 = e^0\cdot\mu e^0 = \mu = E(X)$

$$M''(t) = e^{t+(e^t-1)}\cdot\mu(1+\mu e^t)$$

so $M''(0) = e^{0+(1-1)}\cdot\mu(1+\mu e^0)$

$$= e^0\cdot\mu(1+\mu) = \mu+\mu^2 = E(X^2)$$

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① $X \sim \text{Geo}(p)$ $\qquad$ $P(X=k) = (1-p)^{k-1}p$

$$M_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p$$

$$= \underset{k=1}{e^t(1-p)^0p} \xrightarrow{\times e^t(1-p)} + \underset{k=2}{e^{2t}(1-p)^1p} \xrightarrow{\times e^t(1-p)} + \underset{k=3}{e^{3t}(1-p)^2p} + \cdots$$

$\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$

geometric series $\quad$ common ratio: $e^t(1-p) = r$

initial term: $e^t(1-p)^0p = pe^t = a$

$$\frac{a}{1-r} \to \quad = \frac{pe^t}{1-e^t(1-p)} = M_X(t)$$