1. Suppose that 45% of the phone calls you receive are scam calls. Assume that the probability of a scam call is independent from one call to the next.

(a) Let $X = 1$ if the next call you receive is from a scam call, and $X = 0$ otherwise. What type of random variable is X? What are its mean and standard deviation?

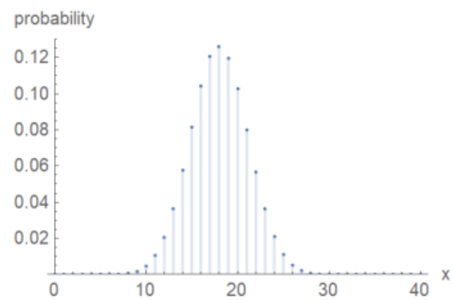
$X \sim$ Bernoulli with $p = 0.45$, or equivalently $X \sim \text{Bin}(1, 0.45)$.

$E(X) = 0.45$, $\sigma_X = \sqrt{(0.45)(0.55)} = 0.497$

(b) Let Y be the number of scam calls in the next 40 phone calls. What type of random variable is Y? Sketch the pmf of Y.

$Y \sim \text{Bin}(40, 0.45)$

pmf:



(c) What are the mean and standard deviation of Y?

$E(Y) = 40(0.45) = 18$

$\sigma_Y = \sqrt{40(0.45)(0.55)} = 3.14$

(d) Suppose that you lose 30 seconds of your time every time a scammer calls your phone. What are the expected value and standard deviation of the amount of time you will lose over the next 40 phone calls?

Let $Z = 30Y$ be the number of seconds you lose.

Then $E(Z) = 30E(Y) = 540$ seconds, and $\sigma_Z = 30\sigma_Y = 94$ seconds.

2. A coin that lands on heads with probability p is flipped ten times. Given that a total of 6 heads results, what is the conditional probability that the first three flips are heads, tails, heads (in that order)?

Let $X \sim \text{Bin}(10, p)$ be the number of heads in all ten flips.

Let $Y \sim \text{Bin}(7, p)$ be the number of heads in the last 7 flips

Then:

$$P(HTH \mid X=6) = \frac{P(HTH \cap X=6)}{P(X=6)} = \frac{P(HTH)\,P(Y=4)}{P(X=6)}$$

$$= \frac{p^2(1-p)\cdot\binom{7}{4}p^4(1-p)^3}{\binom{10}{6}p^6(1-p)^4} = \frac{35}{210} = \boxed{\frac{1}{6}}$$

3. Among persons donating blood to a clinic, 85% have Rh$^+$ blood. Six people donate blood at the clinic on a particular day.

(a) Find the probability that at most three of the six have Rh$^+$ blood.

$X \sim \text{Bin}(6, 0.85)$ $\qquad$ $P(X \leq 3) = B(3; 6, 0.85) = 0.047$

(b) Find the probability that at most one of the six does not have Rh$^+$ blood.

$$P(X \geq 5) = b(5; 6, 0.85) + b(6; 6, 0.85) = 0.776$$

or: $= 1 - B(4; 6, 0.85)$

(c) What is the probability that the number of Rh$^+$ donors lies within two standard deviations of the mean number?

$E(X) = 5.1$, $\qquad \sigma_X = 0.875$

$$P(3.35 < X < 6.85) = P(X=4) + P(X=5) + P(X=6) = 0.953$$

Note: Chebyshev's Inequality implies $P(|X-\mu| < 2\sigma) \geq \frac{3}{4}$, which is true, but the pmf gives a better answer.

(d) The clinic needs six Rh$^+$ donors on a certain day. How many people must donate blood to have the probability of obtaining blood from at least six Rh$^+$ donors over 0.95?

Let $Y_n \sim \text{Bin}(n, 0.85)$.

We want n such that $P(Y_n \geq 6) \geq 0.95$.

Testing some n, we find:

$$P(Y_8 \geq 6) = 0.895 \quad \text{and} \quad P(Y_9 \geq 6) = 0.966$$

Thus, the clinic needs at least 9 blood donors.

BONUS: A system consists of n components, each of which will independently function with probability p. The system will operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate than a 3-component system?

Let $X \sim Bin(5, p)$. The probability that a 5-component system functions effectively is $P(X \geq 3)$.

Similarly, let $Y \sim Bin(3, p)$. The probability that a 3-component system functions effectively is $P(Y \geq 2)$.

Thus, we want p such that:

$$P(X \geq 5) > P(Y \geq 2)$$

$$P(X=3) + P(X=4) + P(X=5) > P(Y=2) + P(Y=3)$$

$$10\,p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

Simplify to obtain:

$$3(p-1)^2(2p-1) > 0$$

$$p > \frac{1}{2}$$

A 5-component system is more likely than a 3-component system to operate effectively if $p > \frac{1}{2}$.