1. Let X be a random variable with pmf given by $p(4) = 0.3$, $p(5) = 0.2$, $p(8) = 0.3$, $p(10) = 0.2$.

(a) What is the expected value $E(X)$?

$$E(X) = 4(0.3) + 5(0.2) + 8(0.3) + 10(0.2) = 6.6$$

(b) What is $E(X^2)$?

$$E(X^2) = 4^2(0.3) + 5^2(0.2) + 8^2(0.3) + 10^2(0.2) = 49$$

(c) What is $\text{Var}(X)$? *Hint: use the shortcut formula!*

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 49 - 6.6^2 = 5.44$$

(d) Suppose the random variable is part of a game in which you win $2X - 8$ dollars. Let $Y = 2X - 8$. What is the pmf of Y?

y	0	2	8	12
$p_Y(y)$	0.3	0.2	0.3	0.2

(e) Use the pmf of Y to find $E(Y)$, your expected winnings in this game.

$$E(Y) = 0(0.3) + 2(0.2) + 8(0.3) + 12(0.2) = 5.2$$

(f) Use the pmf of Y to find $E(Y^2)$, and then find $\text{Var}(Y)$.

$$E(Y^2) = 0^2(0.3) + 2^2(0.2) + 8^2(0.3) + 12^2(0.2) = 48.8$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = 48.8 - (5.2)^2 = 21.76$$

(g) How is $E(Y)$ related to $E(X)$? How is $\text{Var}(Y)$ related to $\text{Var}(X)$? $\cancel{\text{Var}(Y) = 2\,\text{Var}(X) - 8}$

$Y = 2X - 8$

$E(Y) = 2E(X) - 8$ and $\text{Var}(Y) = 2^2\,\text{Var}(X)$

Expected value is linear, but variance is not!

Expected value is linear, but variance is not!

$$E(aX+b) = aE(X)+b$$

$$E(af(X)+bg(X)+c) = aE(f(X)) + bE(g(X)) + c$$

$$Var(aX+b) = a^2\,Var(X)$$

$$\sigma_{aX+b} = |a|\,\sigma_X$$

why? $Var(aX+b) = E((aX+b)^2) - E(aX+b)^2$

$= E(a^2X^2 + 2abX + b^2) - (aE(X)+b)^2$

$= \cdots = a^2(E(X^2) - E(X)^2) = a\,Var(X)$

Chebyshev's Inequality: Let X be a discrete random variable with mean μ and standard deviation σ. For any $k \geq 1$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

In words, the probability that X is at least k standard deviations away from its mean is at most $\frac{1}{k^2}$.

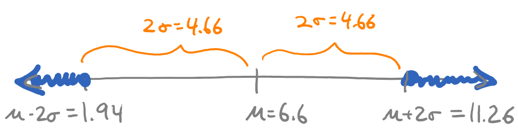
2. Verify that Chebyshev's Inequality holds for the random variable X from Problem 1, using the value $k = 2$. That is, check that $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

Note that $\sigma_X = \sqrt{5.44} \approx 2.33$

Thus, we have: $\mu = 6.6$, $\sigma = 2.33$, and $k = 2$.

Consider: $P(|X - 6.6| \geq 2(2.33))$

$= P(|X - 6.6| \geq 4.66)$



$= P(X \leq 1.94 \text{ or } X \geq 11.26)$

$= 0$

Since $P(|X-\mu| \geq 2\sigma) = 0$, which is less than $\frac{1}{2^2} = \frac{1}{k^2}$,

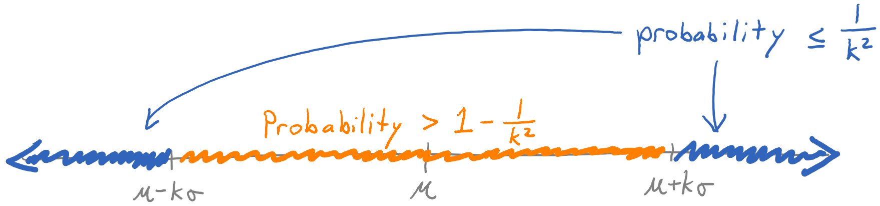
we see that Chebyshev's inequality holds in this case.

3. The number of equipment breakdowns in a manufacturing plant averages 4 per week, with standard deviation 0.7 per week.

(a) Find an interval that includes at least 90% of the weekly figures for the number of breakdowns.

Apply Chebyshev's Inequality with k that solves $\frac{1}{k^2} = 0.1$.

That is $k = \sqrt{10} \approx 3.16$.



We want: $1 - \frac{1}{k^2} \geq 0.9$

Chebyshev's Inequality then says:

$$P(|X-4| \geq 3.16(0.7)) = P(|X-4| \geq 2.21)$$
$$= P(X \leq 1.79 \text{ or } X \geq 6.21) \leq 0.1$$

Take the complement to flip the inequality:

$$P(1.79 < X < 6.21) > 0.9$$

So the interval $(1.79, 6.21)$ includes at least 90% of the numbers of weekly breakdowns.

(b) A plant supervisor promises that the number of breakdowns will rarely exceed 7 in a one-week period. Is the supervisor justified in making this claim? Why?

From part (a), we see that 90% of weeks have less than 7 breakdowns.

We can do even better if we apply Chebyshev's Inequality with $k=5$:

$$P(|X-4| \geq 5(0.7)) = P(X \leq 0.5 \text{ or } X \geq 7.5) = P(X=0) + P(X>7) \leq \frac{1}{5^2}$$

So $P(X > 7) \leq \frac{1}{25} = 0.04$.

Thus, the probability of more than 7 breakdowns in a week is not greater than 0.04. The supervisor's claim seems justified.

BONUS: When flipped, a certain coin comes up heads with probability p. Let X be the number of heads in n flips of this coin.

(a) What is the probability distribution of X?

$$P(X=x) = \underbrace{\binom{n}{x}}_{\text{number of sequences of } x \text{ heads and } n-x \text{ tails}} \underbrace{p^x (1-p)^{n-x}}_{\text{probability of any particular sequence of } x \text{ heads and } n-x \text{ tails}} \quad \text{for } x = 0, 1, 2, \dots, n$$

(b) Show that $E(X) = np$.

$$E(X) = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} x \frac{(n-1)!}{x!\,(n-x)!} p^{x-1}(1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1}(1-p)^{n-x}$$

let $j = x-1$

$$= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!\,(n-j-1)!} p^{j} (1-p)^{n-j-1}$$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1-p)^{n-j-1}$$

binomial theorem

$$= np \left(p + (1-p)\right)^{n-j-1}$$

$$= np \,(1)^{n-j-1} = np$$